A Flag-Manifold Balancing Method for Convex Decomposition of Quantum Channels

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Abstract

Quantum channels of fixed input and output dimensions form a high-dimensional compact convex set. Every channel can therefore be represented as a convex combination of extreme channels, but the general bound obtained from Carathéodory's theorem is typically much larger than what is suggested by the rank structure of the channel. The Ruskai–Audenaert conjecture [3] proposes that a channel from a d-dimensional system to a d-dimensional system can be written as a convex combination of only d generalized extreme channels, each having Choi rank at most d.

Motivated by the recent breakthrough [9], in this work we describe a target-adapted numerical framework based on the partial complex flag manifold

$$U(d^2)/U(d)^d.$$

For a target Choi matrix J, we parameterize a d-term positive low-rank decomposition by

$$P_\mu = J^{1/2} U \Pi_\mu U^\dagger J^{1/2},$$

where the Π_μ are fixed mutually orthogonal projections of rank d. This parameterization automatically guarantees positivity, the rank bound, and the exact identity

$$\sum_{\mu=1}^{d} P_\mu = J.$$

The remaining numerical task is only to balance the partial traces of the components. Once this balancing problem is solved, the convex weights and generalized extreme channels are obtained automatically.

Each resulting generalized extreme channel is subsequently converted to a Kraus representation and compiled using a recursive thin CSD. The output consists of the CSD angles, the initial system unitary, the intermediate conditional unitaries, the branch-dependent output unitaries, and optional two-level complex Givens gate parameters.

Contents

1 Background

1.1 The convex set of quantum channels

Let the input and output Hilbert spaces both be $\mathbb{C}^d$. A quantum channel is a completely positive and trace-preserving linear map

$$\Phi : \mathcal{L}(\mathbb{C}^d) \longrightarrow \mathcal{L}(\mathbb{C}^d).$$

It admits a Kraus representation

$$\Phi(\rho) = \sum_{a=1}^{r} K_a \rho K_a^\dagger, \qquad \sum_{a=1}^{r} K_a^\dagger K_a = \mathbb{I}_d. \tag{1}$$

The smallest possible number r of Kraus operators is called the Kraus rank of the channel.

The collection of all channels with fixed input and output dimensions is a compact convex set. If Φ_1 and Φ_2 are channels and $0 \leq p \leq 1$, then

$$p\Phi_1 + (1-p)\Phi_2$$

is again a channel.

A channel Φ is called an extreme channel if it cannot be written as a nontrivial convex combination of two distinct channels. Choi's extremality criterion states that if (1) is a minimal Kraus representation, then

$$\Phi \text{ is extreme} \quad \Longleftrightarrow \quad \left\{K_a^\dagger K_b\right\}_{a,b=1}^{r} \text{ is linearly independent in } M_d(\mathbb{C}). \tag{2}$$

Since $M_d(\mathbb{C})$ has complex dimension d^2, an extreme channel necessarily satisfies

$$r \leq d.$$

1.2 The Choi representation

We use the unnormalized Choi matrix

$$J_\Phi = \sum_{i,j=1}^{d} |i\rangle\langle j| \otimes \Phi(|i\rangle\langle j|). \tag{3}$$

The ordering of tensor factors is chosen to be

$$\mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{out}}.$$

For a $d \times d$ Kraus operator K, MATLAB column vectorization $K(:)$ corresponds to

$$|K\rangle\!\rangle = \sum_{i=1}^{d} |i\rangle_{\text{in}} \otimes K|i\rangle_{\text{out}}.$$

The Choi matrix can therefore be written as

$$J_\Phi = \sum_{a=1}^{r} |K_a\rangle\!\rangle\langle\!\langle K_a|. \tag{4}$$

Complete positivity and trace preservation are equivalent to

$$J_\Phi \geq 0, \qquad \operatorname{Tr}_{\text{out}} J_\Phi = \mathbb{I}_d. \tag{5}$$

Moreover,

$$\operatorname{rank} J_\Phi = \text{Kraus rank of } \Phi. \tag{6}$$

If

$$J_\Phi = \sum_{a=1}^{r} \lambda_a |v_a\rangle\langle v_a|, \qquad \lambda_a > 0,$$

then a set of Kraus operators can be recovered as

$$K_a = \operatorname{unvec}\left(\sqrt{\lambda_a}|v_a\rangle\right). \tag{7}$$

1.3 Generalized extreme channels

The set of extreme channels is generally not closed. The closure of the set of extreme channels is precisely the set of channels whose Choi rank does not exceed the input dimension:

$$\overline{\mathcal{E}_d} = \{\Phi : \operatorname{rank} J_\Phi \le d\} \,. \tag{8}$$

These channels are called generalized extreme channels.

A generalized extreme channel can be either:

(i) a genuine extreme channel satisfying (2); or

(ii) a channel of Choi rank at most d that fails the linear independence condition, often called a quasi-extreme channel.

The Ruskai–Audenaert decomposition problem concerns generalized extreme channels, not necessarily genuine extreme channels.

2 The Ruskai–Audenaert Decomposition Problem

2.1 Weak and strong forms

For a channel

$$\Phi : M_d(\mathbb{C}) \longrightarrow M_d(\mathbb{C}),$$

the weak Ruskai–Audenaert decomposition asks whether there exist d generalized extreme channels Φ_μ^{g} and probabilities p_μ such that

$$\Phi = \sum_{\mu=1}^{d} p_\mu \Phi_\mu^{\mathrm{g}}, \qquad p_\mu \ge 0, \qquad \sum_{\mu=1}^{d} p_\mu = 1, \tag{9}$$

with

$$\operatorname{rank} J_{\Phi_\mu^{\mathrm{g}}} \le d. \tag{10}$$

The strong form additionally requires equal weights:

$$p_\mu = \frac{1}{d}, \qquad \Phi = \frac{1}{d}\sum_{\mu=1}^{d} \Phi_\mu^{\mathrm{g}}. \tag{11}$$

In the Choi representation, define the weighted positive components

$$P_\mu = p_\mu J_{\Phi_\mu^{\mathrm{g}}}. \tag{12}$$

The weak decomposition is therefore equivalent to finding

$$J_\Phi = \sum_{\mu=1}^{d} P_\mu, \qquad P_\mu \ge 0, \qquad \operatorname{rank} P_\mu \le d, \tag{13}$$

such that

$$\operatorname{Tr}_{\mathrm{out}} P_\mu = p_\mu \mathbb{I}_d. \tag{14}$$

The strong form is equivalent to

$$\operatorname{Tr}_{\mathrm{out}} P_\mu = \frac{1}{d}\mathbb{I}_d. \tag{15}$$

2.2 The rank lower bound on the number of terms

Suppose that J_Φ has full rank,

$$\operatorname{rank} J_\Phi = d^2.$$

If every positive component has rank at most d, then

$$\operatorname{rank}\left(\sum_{\mu=1}^{m} P_\mu\right) \le \sum_{\mu=1}^{m} \operatorname{rank} P_\mu \le md.$$

Consequently,

$$m \ge d.$$

Thus, the d-term decomposition proposed by the Ruskai–Audenaert conjecture reaches the smallest possible number of terms allowed by the rank bound for a full-rank channel.

3 Earlier Numerical Approaches

3.1 Independent parameterization of the components

The numerical approach of Wang and Sanders [4] approximates a target channel by

$$J_\Phi \approx \sum_{\mu=1}^{d} p_\mu J_\mu^{\mathrm{g}}, \tag{16}$$

where each J_μ^{g} is independently parameterized as the Choi matrix of a generalized extreme channel.

The optimization variables include:

1. the convex weights p_μ;
2. the parameters of d generalized extreme channels;
3. unitary pre- and post-processing matrices;
4. multiplexer rotation angles;
5. additional internal unitaries in the full CSD construction.

The optimization objective is usually a distance between the target Choi matrix and the Choi matrix of the candidate mixture.

3.2 Recursive CSD parameterization

Wang [5] subsequently used a recursive cosine–sine decomposition to construct a more complete parameterization of generalized extreme channels.

For a qutrit channel, the Kraus operators can be written as

$$\begin{aligned} K_0 &= W_0 C_0 V, \\ K_1 &= W_1 C_1 Z_1 S_0 V, \\ K_2 &= W_2 S_1 Z_1 S_0 V, \end{aligned} \tag{17}$$

where V, Z_1, W_0, W_1, W_2 are unitary matrices and

$$C_j = \operatorname{diag}(\cos\theta_{j1}, \ldots, \cos\theta_{jd}),$$

$$S_j = \operatorname{diag}(\sin\theta_{j1}, \ldots, \sin\theta_{jd}).$$

This approach has several advantages:

1. every candidate component is completely positive;
2. every candidate component is trace preserving;
3. every candidate component has Choi rank at most d;
4. the parameters can be converted directly into a quantum circuit.

However, in the global convex-decomposition problem, the equality

$$\sum_{\mu=1}^{d} p_\mu J_\mu^{\mathrm{g}} = J_\Phi$$

is imposed only through the fitting objective. Moreover, the parameterization contains a substantial amount of gauge redundancy.

4 Target-Adapted Flag-Manifold Parameterization

4.1 Reference projections

Let

$$N = d^2.$$

Choose d mutually orthogonal projections on $\mathbb{C}^N$,

$$\Pi_\mu \Pi_\nu = \delta_{\mu\nu}\Pi_\mu, \qquad \operatorname{rank}\Pi_\mu = d, \qquad \sum_{\mu=1}^{d} \Pi_\mu = \mathbb{I}_N. \tag{18}$$

A convenient standard choice is

$$\Pi_\mu = \sum_{k=(\mu-1)d+1}^{\mu d} |k\rangle\langle k|.$$

For $U \in U(N)$, define

$$Q_\mu(U) = U \Pi_\mu U^\dagger. \tag{19}$$

Then

$$Q_\mu Q_\nu = \delta_{\mu\nu} Q_\mu, \qquad \operatorname{rank} Q_\mu = d, \qquad \sum_{\mu=1}^{d} Q_\mu = \mathbb{I}_N.$$

Right multiplication by a block-diagonal unitary,

$$U \longmapsto U\,\mathrm{diag}(V_1, \ldots, V_d), \qquad V_\mu \in U(d),$$

does not change any of the Q_μ. The true parameter space is therefore the partial complex flag manifold

$$\boxed{\mathcal{F}_d(\mathbb{C}^{d^2}) \simeq U(d^2)/U(d)^d.} \tag{20}$$

Its real dimension is

$$\dim_{\mathbb{R}} \mathcal{F}_d(\mathbb{C}^{d^2}) = d^4 - d^3 = d^3(d-1). \tag{21}$$

For $d = 3$,

$$\dim_{\mathbb{R}} U(9)/U(3)^3 = 81 - 27 = 54.$$

4.2 Positive low-rank components from the target Choi matrix

Let J be the target Choi matrix in the unnormalized convention

$$J \geq 0, \qquad \mathrm{Tr}_{\mathrm{out}} J = \mathbb{I}_d.$$

Set

$$S = J^{1/2}.$$

For every $U \in U(d^2)$, define

$$\boxed{P_\mu(U) = SQ_\mu(U)S = J^{1/2} U \Pi_\mu U^\dagger J^{1/2}.} \tag{22}$$

Proposition 4.1. *For every $U \in U(d^2)$, the components in (22) satisfy*

$$P_\mu(U) \geq 0, \qquad \mathrm{rank}\, P_\mu(U) \leq d, \tag{23}$$

and

$$\sum_{\mu=1}^{d} P_\mu(U) = J. \tag{24}$$

Proof. Since $Q_\mu \geq 0$,

$$P_\mu = SQ_\mu S \geq 0.$$

Moreover,

$$\mathrm{rank}\, P_\mu \leq \mathrm{rank}\, Q_\mu = d.$$

Finally,

$$\sum_{\mu=1}^{d} P_\mu = S \left(\sum_{\mu=1}^{d} Q_\mu \right) S = S \mathbb{I}_{d^2} S = J.$$

□

Thus, positivity, the rank constraint, and exact reconstruction of the target Choi matrix are all built into the parameterization.

4.3 Completeness for full-rank target Choi matrices

The flag-manifold formula is not merely a convenient ansatz. For a full-rank target J, it parameterizes every saturated d-term decomposition into components of rank at most d.

Proposition 4.2. *Let $J > 0$, and suppose that*

$$J = \sum_{\mu=1}^{d} P_\mu, \qquad P_\mu \geq 0, \qquad \mathrm{rank}\, P_\mu \leq d.$$

Then there exist mutually orthogonal rank-d projections Q_μ such that

$$P_\mu = J^{1/2} Q_\mu J^{1/2}.$$

Equivalently, there exists $U \in U(d^2)$ such that

$$Q_\mu = U \Pi_\mu U^\dagger.$$

Proof. Define

$$Q_\mu = J^{-1/2} P_\mu J^{-1/2}.$$

Then

$$Q_\mu \geq 0, \qquad \sum_{\mu=1}^{d} Q_\mu = \mathbb{I}_{d^2}, \qquad \operatorname{rank} Q_\mu \leq d.$$

The rank inequalities imply

$$d^2 = \operatorname{rank} \mathbb{I}_{d^2} \leq \sum_{\mu=1}^{d} \operatorname{rank} Q_\mu \leq d^2.$$

Therefore every inequality is saturated and

$$\operatorname{rank} Q_\mu = d.$$

Write

$$Q_\mu = Y_\mu Y_\mu^\dagger, \qquad Y_\mu \in \mathbb{C}^{d^2 \times d},$$

and form the square matrix

$$Y = \begin{pmatrix} Y_1 & Y_2 & \cdots & Y_d \end{pmatrix} \in \mathbb{C}^{d^2 \times d^2}.$$

Since

$$YY^\dagger = \sum_{\mu=1}^{d} Y_\mu Y_\mu^\dagger = \mathbb{I}_{d^2},$$

the square matrix Y is unitary. Hence

$$Y^\dagger Y = \mathbb{I}_{d^2},$$

so the columns belonging to different Y_μ are mutually orthogonal. Consequently, each $Q_\mu = Y_\mu Y_\mu^\dagger$ is an orthogonal projection and the projections are mutually orthogonal. □

Remark 4.3. *If J is singular, the parameterization (22) remains valid and useful. However, completeness must then be formulated on the support of J, possibly using an isometric embedding of* $\operatorname{supp} J$ *into* $\mathbb{C}^{d^2}$*.*

5 Partial-Trace Balancing

5.1 The weak objective

Define

$$A_\mu(U) = \operatorname{Tr}_{\text{out}} P_\mu(U) \in \operatorname{Herm}(\mathbb{C}^d). \tag{25}$$

Because

$$\sum_{\mu=1}^{d} P_\mu = J$$

and

$$\operatorname{Tr}_{\text{out}} J = \mathbb{I}_d,$$

we have

$$\sum_{\mu=1}^{d} A_\mu(U) = \mathbb{I}_d. \tag{26}$$

Let

$$\mathcal{T}_0(X) = X - \frac{\operatorname{Tr} X}{d}\mathbb{I}_d \tag{27}$$

denote projection onto the traceless Hermitian subspace.

The weak balancing objective is

$$\boxed{F_{\text{weak}}(U) = \sum_{\mu=1}^{d} \left\| A_\mu(U) - \frac{\operatorname{Tr} A_\mu(U)}{d}\mathbb{I}_d \right\|_{\text{F}}^2.} \tag{28}$$

The condition

$$F_{\text{weak}}(U) = 0$$

is equivalent to

$$A_\mu(U) = p_\mu \mathbb{I}_d$$

for every μ, where

$$\boxed{p_\mu = \frac{\operatorname{Tr} A_\mu}{d} = \frac{\operatorname{Tr} P_\mu}{d}.} \tag{29}$$

Equation (26) then implies

$$p_\mu \geq 0, \qquad \sum_{\mu=1}^{d} p_\mu = 1.$$

For every $p_\mu > 0$, define

$$J_\mu^{\text{g}} = \frac{P_\mu}{p_\mu}. \tag{30}$$

Then

$$J = \sum_{\mu=1}^{d} p_\mu J_\mu^{\text{g}},$$

and

$$J_\mu^{\text{g}} \geq 0, \qquad \operatorname{Tr}_{\text{out}} J_\mu^{\text{g}} = \mathbb{I}_d, \qquad \operatorname{rank} J_\mu^{\text{g}} \leq d.$$

Thus J_μ^{g} is the Choi matrix of a generalized extreme channel.

5.2 The strong objective

For the strong decomposition, each component must satisfy

$$A_\mu = \frac{1}{d}\mathbb{I}_d.$$

The corresponding objective is

$$\boxed{F_{\text{strong}}(U) = \sum_{\mu=1}^{d} \left\| A_\mu(U) - \frac{1}{d}\mathbb{I}_d \right\|_{\text{F}}^2.} \tag{31}$$

The weak and strong objectives satisfy

$$F_{\text{strong}} = F_{\text{weak}} + d\sum_{\mu=1}^{d} \left(p_\mu - \frac{1}{d} \right)^2. \tag{32}$$

Thus, the strong problem consists of the weak partial-trace balancing problem together with the additional requirement that all weights be equal.

5.3 Number of independent equations

The real vector space of traceless $d \times d$ Hermitian matrices has dimension

$$d^2 - 1.$$

Since

$$\sum_{\mu=1}^{d} \mathcal{T}_0(A_\mu) = 0,$$

only $d-1$ of the d traceless residuals are independent. Therefore, the weak balancing problem contains

$$(d-1)(d^2-1) \tag{33}$$

independent real equations.

The real dimension of the flag manifold is

$$d^3(d-1).$$

The difference is

$$d^3(d-1) - (d-1)(d^2-1) = (d-1)(d^3 - d^2 + 1) > 0.$$

Hence, from a local dimension-counting viewpoint, the weak balancing problem is underdetermined. This observation alone, however, does not imply global existence of a zero.

6 Riemannian Gradient on the Flag Manifold

6.1 Variations on the unitary group

Let X be anti-Hermitian,

$$X^\dagger = -X,$$

and consider the curve

$$U(t) = e^{tX} U. \tag{34}$$

Since

$$Q_\mu = U \Pi_\mu U^\dagger,$$

we obtain

$$\left.\frac{dQ_\mu}{dt}\right|_{t=0} = [X, Q_\mu]. \tag{35}$$

For the weak objective, define

$$H_\mu = A_\mu - \frac{\operatorname{Tr} A_\mu}{d} \mathbb{I}_d \tag{36}$$

and

$$R_\mu = S\,(H_\mu \otimes \mathbb{I}_d)\,S. \tag{37}$$

Here the tensor factor ordering is $\mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{out}}$. With the opposite Choi convention, the two tensor factors in (37) must be interchanged.

The first variation of the objective is

$$\begin{aligned}\left.\frac{dF_{\text{weak}}}{dt}\right|_{t=0} &= 2\sum_{\mu=1}^{d} \operatorname{Tr}\left[H_\mu \frac{dA_\mu}{dt}\right] \\ &= 2\sum_{\mu=1}^{d} \operatorname{Tr}\left[R_\mu[X, Q_\mu]\right] \\ &= 2\operatorname{Tr}\left[X\sum_{\mu=1}^{d}[Q_\mu, R_\mu]\right]. \end{aligned} \tag{38}$$

Define the anti-Hermitian generator

$$\boxed{G(U) = \sum_{\mu=1}^{d}[Q_\mu, R_\mu].} \tag{39}$$

Choosing $X = G$ gives

$$\left.\frac{dF_{\text{weak}}}{dt}\right|_{t=0} = -2\,\|G\|_{\mathrm{F}}^2\,. \tag{40}$$

A structure-preserving descent step is therefore

$$\boxed{U_{k+1} = e^{\eta_k G(U_k)}U_k,} \tag{41}$$

where the step size $\eta_k > 0$ may be selected by an Armijo backtracking line search.

6.2 Gauge directions and the horizontal tangent space

The parameterization is invariant under

$$U \sim U\,\mathrm{diag}(V_1, \dots, V_d), \qquad V_\mu \in U(d).$$

These block-diagonal directions are gauge directions.

Relative to the reference block decomposition, a horizontal anti-Hermitian tangent vector can be written as

$$X = \begin{pmatrix} 0 & X_{12} & \cdots & X_{1d} \\ -X_{12}^\dagger & 0 & \cdots & X_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ -X_{1d}^\dagger & -X_{2d}^\dagger & \cdots & 0 \end{pmatrix}, \tag{42}$$

where every $X_{\mu\nu} \in M_d(\mathbb{C})$.

The commutator form in (39) automatically removes the block-diagonal gauge component.

7 Extraction of Generalized Extreme Channels

7.1 Raw normalized components

Once the optimization has produced components P_μ, define

$$p_\mu = \frac{\operatorname{Tr} P_\mu}{d}.$$

The raw normalized components are

$$J_\mu^{\text{raw}} = \frac{P_\mu}{p_\mu}. \tag{43}$$

They satisfy the exact reconstruction identity

$$J = \sum_{\mu=1}^{d} p_\mu J_\mu^{\mathrm{raw}}. \tag{44}$$

If the balancing objective has not converged exactly to zero, the raw components may only approximately satisfy

$$\mathrm{Tr}_{\mathrm{out}}\, J_\mu^{\mathrm{raw}} \approx \mathbb{I}_d.$$

7.2 Local trace-preserving correction

Let

$$A_\mu = \mathrm{Tr}_{\mathrm{out}}\, P_\mu.$$

If $A_\mu > 0$, define

$$\widehat{J}_\mu = \left(A_\mu^{-1/2} \otimes \mathbb{I}_d\right) P_\mu \left(A_\mu^{-1/2} \otimes \mathbb{I}_d\right). \tag{45}$$

Then

$$\mathrm{Tr}_{\mathrm{out}}\, \widehat{J}_\mu = \mathbb{I}_d \tag{46}$$

holds exactly, while

$$\widehat{J}_\mu \geq 0, \qquad \mathrm{rank}\, \widehat{J}_\mu \leq d.$$

At an exact weak balancing point,

$$A_\mu = p_\mu \mathbb{I}_d,$$

and therefore

$$\widehat{J}_\mu = \frac{P_\mu}{p_\mu}.$$

At finite numerical precision, local correction can slightly disturb the exact reconstruction identity. It is therefore important to report both:

1. the exact-sum error of the raw positive components;
2. the trace-preserving error of the raw normalized components;
3. the trace-preserving error after correction;
4. the mixture error after correction.

7.3 Kraus operators and the extremality test

Diagonalize the corrected component:

$$\widehat{J}_\mu = \sum_{a=1}^{r_\mu} \lambda_{\mu a} |v_{\mu a}\rangle\langle v_{\mu a}|, \qquad r_\mu \leq d.$$

The Kraus operators are

$$K_{\mu a} = \mathrm{unvec}\left(\sqrt{\lambda_{\mu a}} |v_{\mu a}\rangle\right). \tag{47}$$

They satisfy

$$\sum_{a=1}^{r_\mu} K_{\mu a}^\dagger K_{\mu a} = \mathbb{I}_d.$$

To test whether the generalized extreme component is a genuine extreme channel, form the matrix

$$M_\mu = \begin{pmatrix} \mathrm{vec}(K_{\mu 1}^\dagger K_{\mu 1}) & \mathrm{vec}(K_{\mu 1}^\dagger K_{\mu 2}) & \cdots & \mathrm{vec}(K_{\mu r_\mu}^\dagger K_{\mu r_\mu}) \end{pmatrix}. \tag{48}$$

The channel is extreme if and only if

$$\mathrm{rank}\, M_\mu = r_\mu^2. \tag{49}$$

8 Recursive Thin CSD and Circuit Parameters

8.1 The Stinespring isometry

Pad the Kraus representation with zero matrices if necessary so that it contains exactly d Kraus operators. Stack them vertically:

$$V_{\text{iso}} = \begin{pmatrix} K_0 \\ K_1 \\ \vdots \\ K_{d-1} \end{pmatrix} \in \mathbb{C}^{d^2 \times d}. \tag{50}$$

The trace-preserving condition implies

$$V_{\text{iso}}^\dagger V_{\text{iso}} = \sum_{j=0}^{d-1} K_j^\dagger K_j = \mathbb{I}_d.$$

Thus V_{iso} is a Stinespring isometry from $\mathbb{C}^d$ to $\mathbb{C}^d \otimes \mathbb{C}^d$.

8.2 The first thin CSD step

Partition the isometry into its first $d \times d$ block and its remaining rows:

$$V_{\text{iso}} = \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}.$$

Take a singular-value decomposition

$$A_0 = W_0 C_0 R_0, \tag{51}$$

where

$$C_0 = \operatorname{diag}(c_{01}, \ldots, c_{0d}), \qquad 0 \le c_{0k} \le 1.$$

Define

$$S_0 = \sqrt{\mathbb{I}_d - C_0^2} = \operatorname{diag}(s_{01}, \ldots, s_{0d}).$$

Since

$$A_0^\dagger A_0 + B_0^\dagger B_0 = \mathbb{I}_d,$$

we have

$$B_0^\dagger B_0 = R_0^\dagger S_0^2 R_0.$$

Consequently, there exists another isometry Y_1 such that

$$B_0 = Y_1 S_0 R_0. \tag{52}$$

8.3 Recursive form

The same procedure is applied recursively to Y_1. The resulting Kraus representation is

$$\boxed{\begin{aligned} K_0 &= W_0 C_0 R_0, \\ K_1 &= W_1 C_1 R_1 S_0 R_0, \\ K_2 &= W_2 C_2 R_2 S_1 R_1 S_0 R_0, \\ &\vdots \\ K_{d-2} &= W_{d-2} C_{d-2} R_{d-2} S_{d-3} R_{d-3} \cdots S_0 R_0, \\ K_{d-1} &= W_{d-1} S_{d-2} R_{d-2} S_{d-3} R_{d-3} \cdots S_0 R_0. \end{aligned}} \tag{53}$$

Here,

$$C_j = \operatorname{diag}(\cos\theta_{j1}, \ldots, \cos\theta_{jd}), \tag{54}$$

and

$$S_j = \operatorname{diag}(\sin\theta_{j1}, \ldots, \sin\theta_{jd}), \tag{55}$$

with

$$0 \le \theta_{jk} \le \frac{\pi}{2}.$$

In notation similar to Wang's construction,

$$V = R_0, \qquad Z_j = R_j, \quad j = 1, \ldots, d-2. \tag{56}$$

For $d = 3$, equation (53) becomes

$$\begin{aligned} K_0 &= W_0 C_0 V, \\ K_1 &= W_1 C_1 Z_1 S_0 V, \\ K_2 &= W_2 S_1 Z_1 S_0 V, \end{aligned}$$

which agrees with the recursive qutrit CSD form.

For $d = 4$, the Kraus operators are

$$\begin{aligned} K_0 &= W_0 C_0 V, \\ K_1 &= W_1 C_1 Z_1 S_0 V, \\ K_2 &= W_2 C_2 Z_2 S_1 Z_1 S_0 V, \\ K_3 &= W_3 S_2 Z_2 S_1 Z_1 S_0 V. \end{aligned}$$

8.4 Multiplexer rotation angles

If each CSD layer is implemented by a two-level rotation $R_y(\varphi)$, the physical multiplexer rotation angle is

$$\boxed{\varphi_{jk} = 2\theta_{jk}.} \tag{57}$$

The MATLAB implementation therefore returns:

- `theta(j,k)`: the CSD half-angle θ_{jk};
- `RyAngles(j,k)`: the rotation angle $2\theta_{jk}$;
- `V`: the initial unitary R_0;
- `Z{j}`: the intermediate unitary R_j;
- `W{j}`: a branch-dependent output unitary.

8.5 Complex Givens decomposition

Every $U(d)$ factor can be further decomposed into two-level complex Givens rotations and a final diagonal phase gate:

$$U = H_1 H_2 \cdots H_M D. \tag{58}$$

Each H_ℓ acts nontrivially only on a two-dimensional subspace and has the block form

$$H_\ell = \begin{pmatrix} \alpha_\ell & -\overline{\beta_\ell} \\ \beta_\ell & \overline{\alpha_\ell} \end{pmatrix}, \tag{59}$$

where

$$\alpha_\ell = e^{i\phi_{\alpha,\ell}} \cos\vartheta_\ell, \qquad \beta_\ell = e^{i\phi_{\beta,\ell}} \sin\vartheta_\ell.$$

Using

$$R_z(\phi) = \mathrm{diag}(e^{-i\phi/2}, e^{i\phi/2}),$$

we obtain

$$H_\ell = R_z(\phi_{\beta,\ell} - \phi_{\alpha,\ell}) R_y(2\vartheta_\ell) R_z(-\phi_{\alpha,\ell} - \phi_{\beta,\ell}). \tag{60}$$

Thus the matrix-valued CSD output can be converted into:

1. pairs of physical levels;
2. R_z phase angles;
3. R_y rotation angles;
4. a final diagonal phase gate.

9 Complete Numerical Procedure

10 MATLAB Output Structure

For the MATLAB command

```
sol = ra_flag_csd(J,d,opts),
```

the main output fields are listed in Table 1.

Table 1: Principal fields of the MATLAB output structure.

Field	Meaning
`sol.p`	Convex weights p_μ.
`sol.P{mu}`	Weighted positive component P_μ, satisfying $\sum_\mu P_\mu = J$ up to roundoff.
`sol.Jraw{mu}`	Raw normalized component P_μ/p_μ.
`sol.Jcomp{mu}`	Locally corrected CPTP generalized extreme Choi matrix.
`sol.Kraus{mu}`	Kraus operators of generalized extreme component μ.
`sol.csd{mu}.theta`	Matrix of recursive CSD half-angles θ_{jk}.
`sol.csd{mu}.RyAngles`	Physical multiplexer rotation angles $2\theta_{jk}$.
`sol.csd{mu}.V`	Initial system unitary $V = R_0$.
`sol.csd{mu}.Z{j}`	Intermediate recursive CSD unitaries $Z_j = R_j$.
`sol.csd{mu}.W{j}`	Branch-dependent output unitaries W_j.
`sol.csd{mu}.gates`	Complex Givens gate decomposition of the unitary factors.
`sol.isExtreme(mu)`	Whether component μ satisfies Choi's genuine extremality criterion.
`sol.objective`	Final value of the flag-manifold objective.
`sol.balanceResidual`	Partial-trace balancing residual $\sqrt{F}$.
`sol.rawSumResidual`	Residual in the identity $\sum_\mu P_\mu = J$.
`sol.mixtureResidual`	Frobenius distance between the target and the corrected CPTP mixture.
`sol.choiTraceDistance`	Trace distance between normalized target and reconstructed Choi states.
`sol.diamondUpperBound`	A conservative diamond-norm upper bound obtained from the Choi trace norm.

Algorithm 1 Flag-manifold channel decomposition followed by recursive CSD

Require: Dimension d, target Choi matrix J, mode mode $\in \{\text{weak}, \text{strong}\}$, and tolerance ε

Ensure: Probabilities $\{p_\mu\}$, generalized extreme Choi matrices, Kraus operators, and recursive CSD parameters

Normalize J so that $\mathrm{Tr}_{\mathrm{out}}\, J = \mathbb{I}_d$
Compute $S = J^{1/2}$
Construct fixed orthogonal rank-d projections $\Pi_1, \dots, \Pi_d$
Initialize $U \in U(d^2)$
while the objective value is larger than ε **do**
  for $\mu = 1, \dots, d$ **do**
    $Q_\mu \leftarrow U \Pi_\mu U^\dagger$
    $P_\mu \leftarrow S Q_\mu S$
    $A_\mu \leftarrow \mathrm{Tr}_{\mathrm{out}}\, P_\mu$
    if mode $=$ weak **then**
      $H_\mu \leftarrow A_\mu - \frac{\mathrm{Tr}\, A_\mu}{d} \mathbb{I}_d$
    else
      $H_\mu \leftarrow A_\mu - \frac{1}{d} \mathbb{I}_d$
    end if
    $R_\mu \leftarrow S(H_\mu \otimes \mathbb{I}_d)S$
  end for
  $G \leftarrow \sum_{\mu=1}^{d} [Q_\mu, R_\mu]$
  Select $\eta > 0$ using Armijo backtracking
  $U \leftarrow e^{\eta G} U$
end while
for $\mu = 1, \dots, d$ **do**
  $p_\mu \leftarrow \mathrm{Tr}(P_\mu)/d$
  $J_\mu^{\mathrm{raw}} \leftarrow P_\mu / p_\mu$
  if finite-precision TP correction is required **then**
    $A_\mu \leftarrow \mathrm{Tr}_{\mathrm{out}}\, P_\mu$

$$J_\mu^{\mathrm{g}} \leftarrow (A_\mu^{-1/2} \otimes \mathbb{I}_d) P_\mu (A_\mu^{-1/2} \otimes \mathbb{I}_d)$$

  else
    $J_\mu^{\mathrm{g}} \leftarrow J_\mu^{\mathrm{raw}}$
  end if
  Diagonalize J_μ^{g} and extract Kraus operators
  Stack the Kraus operators into a Stinespring isometry
  Apply recursive thin CSD
  Record $\{\theta_{jk}, V, Z_j, W_j\}$
  Optionally decompose each unitary into complex Givens gates
end for
return

$$\{p_\mu, J_\mu^{\mathrm{g}}, K_{\mu a}, \theta_{jk}^{(\mu)}, V^{(\mu)}, Z_j^{(\mu)}, W_j^{(\mu)}\}.$$

The final channel representation is

$$\Phi_\mu^{\mathrm{g}}(\rho) = \sum_{a=1}^{r_\mu} K_{\mu a} \rho K_{\mu a}^\dagger, \qquad r_\mu \le d, \tag{61}$$

and

$$\Phi \approx \sum_{\mu=1}^{d} p_\mu \Phi_\mu^{\mathrm{g}}. \tag{62}$$

11 Numerical Diagnostics

Different numerical errors should be reported separately because they arise from different parts of the procedure.

11.1 Flag-manifold balancing error

For the weak problem, define

$$\epsilon_{\mathrm{bal}} = \left[\sum_{\mu=1}^{d} \left\| A_\mu - \frac{\operatorname{Tr} A_\mu}{d} \mathbb{I}_d \right\|_{\mathrm{F}}^2 \right]^{1/2}. \tag{63}$$

This is the direct residual in the weak Ruskai–Audenaert constraints.

11.2 Raw reconstruction error

Since the raw components are generated from a resolution of the identity,

$$\epsilon_{\mathrm{sum}} = \left\| J - \sum_{\mu=1}^{d} P_\mu \right\|_{\mathrm{F}} \tag{64}$$

should be close to machine precision independently of convergence of the balancing objective.

11.3 Trace-preserving residuals

For the raw normalized components,

$$\epsilon_{\mathrm{TP},\mu}^{\mathrm{raw}} = \left\| \operatorname{Tr}_{\mathrm{out}} J_\mu^{\mathrm{raw}} - \mathbb{I}_d \right\|_{\mathrm{F}}. \tag{65}$$

For the corrected components,

$$\epsilon_{\mathrm{TP},\mu}^{\mathrm{comp}} = \left\| \operatorname{Tr}_{\mathrm{out}} J_\mu^{\mathrm{g}} - \mathbb{I}_d \right\|_{\mathrm{F}}. \tag{66}$$

11.4 Corrected CPTP mixture error

After local trace-preserving correction, define

$$\epsilon_{\mathrm{mix}} = \left\| J - \sum_{\mu=1}^{d} p_\mu J_\mu^{\mathrm{g}} \right\|_{\mathrm{F}}. \tag{67}$$

11.5 CSD reconstruction error

Let $\widetilde{K}_{\mu a}$ denote the Kraus operators reconstructed from the recursive CSD factors. Define

$$\epsilon_{\mathrm{CSD},\mu} = \left[\sum_{a=1}^{d} \left\| K_{\mu a} - \widetilde{K}_{\mu a} \right\|_{\mathrm{F}}^{2} \right]^{1/2}. \tag{68}$$

This error tests only the recursive CSD implementation. A small CSD error does not by itself imply that the Ruskai–Audenaert decomposition has been found. The balancing and mixture errors must also be checked.

12 Computational Complexity

The unitary matrix optimized on the flag manifold has size

$$N = d^2.$$

If the update uses a dense matrix exponential

$$e^{\eta G},$$

the cost of one such operation is generally

$$O(N^3) = O(d^6).$$

The storage required for dense $d^2 \times d^2$ Choi matrices, unitaries, and projectors scales as

$$O(d^4).$$

The current implementation is therefore most suitable for:

1. low-dimensional systems such as $d = 2, 3, 4$;
2. numerical investigation of channel-decomposition conjectures;
3. construction of a small number of high-precision decompositions;
4. generation of structured inputs for circuit compilation.

Possible improvements for larger d include:

1. replacing the matrix exponential by a Cayley retraction,

$$U_{k+1} = \left(\mathbb{I} - \frac{\eta}{2} G \right)^{-1} \left(\mathbb{I} + \frac{\eta}{2} G \right) U_k;$$

2. Riemannian conjugate-gradient methods;
3. Riemannian trust-region methods;
4. switching to Gauss–Newton or Levenberg–Marquardt near a root;
5. using low-rank bases for the subspaces instead of storing full projectors;
6. homotopy continuation from a channel with a known decomposition.

13 Conclusion

We have described a numerical framework for decomposing a d-dimensional quantum channel into d candidate generalized extreme channels. The central parameterization is

$$P_\mu = J^{1/2} U \Pi_\mu U^\dagger J^{1/2}, \qquad U \in U(d^2)/U(d)^d.$$

For every value of U, it automatically satisfies

$$P_\mu \geq 0, \qquad \operatorname{rank} P_\mu \leq d, \qquad \sum_{\mu=1}^{d} P_\mu = J.$$

The weak Ruskai–Audenaert problem is reduced to the partial-trace balancing equations

$$\operatorname{Tr}_{\text{out}} P_\mu = p_\mu \mathbb{I}_d.$$

The convex weights arise automatically from

$$p_\mu = \frac{\operatorname{Tr} P_\mu}{d}.$$

After the generalized extreme Choi matrices have been obtained, a recursive thin cosine–sine decomposition produces the circuit parameters

$$\{\theta_{jk}, V, Z_j, W_j\},$$

and all unitary factors may subsequently be decomposed into two-level complex Givens gates.

Compared with the direct optimization of several independently parameterized channels, the flag-manifold formulation has the following structural advantages:

1. the target Choi matrix is decomposed exactly throughout the optimization;
2. positivity and rank constraints require no penalty terms;
3. convex weights are generated automatically;
4. the quotient geometry and gauge freedom are explicit;
5. channel decomposition is separated from circuit compilation;
6. decomposition error and CSD compilation error can be diagnosed independently.

For the weak qutrit problem, the existence of a balancing point is known. For general $d \geq 4$, the method provides a structured numerical platform for investigating the Ruskai–Audenaert conjecture, but numerical convergence alone cannot replace a rigorous proof of existence.

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