Floquet system

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Definition. *A quantum Floquet system is defined by a time-periodic Hamiltonian $H(t) = H(t+T)$ with period T. The evolution within a cycle $U(T)$ can be treated as driven by a time-independent Floquet Hamiltonian F, and in general, $U(t) = P(t)e^{-itF}$ for a time-periodic unitary $P(t) = P(t+T)$.*

Keywords

Magnus expansion; Floquet theorem; Kapitza pendulum; dynamical stabilization; Floquet phases of matter; time crystal; many-body localization; integrability; area law.

Abstract

We introduce the Floquet engineering, which uses time-periodic Hamiltonian drives, in the context of quantum many-body systems. The foundation for dealing with time-periodic evolution is Magnus expansion, which is only physically proper if it converges. The power of Floquet engineering can be seen from a simple example of a classical pendulum under a high-frequency drive, which can change its equilibrate state at will. For quantum many-body systems, care must be paid to avoid heating regime and reach pseudo-equilibrium states, which can support exotic orders of matter, such as time crystal. We then explain the relevant theory of many-body localization, which allows an extensive set of conserved quasi-local observable induced by

strong disorder, and survey some research frontier regarding quantum technology. We conclude with some stories and history of the subject.

1 Minimal version of Floquet theory

1.1 Opening

We might all have such an experience: you enjoy study in a coffee shop but you cannot focus when you are at other places. The background music in coffee shops seems as noises but they are not. They can help us to focus on our study, suppressing random thoughts in our mind or noises in surroundings. This actually can be viewed as kind of dynamical decoupling or Floquet engineering, which uses non-resonant high-frequency drives (the music) to dress the state of our mind to become a robust novel state.

Floquet engineering is widely used in classical physics and engineering. In quantum physics, it has been widely used in quantum chemistry, e.g., to control chemical reactions. Due to the periodicity, Floquet systems can be analytically studied, far more easier than general time-dependent systems. Due to the development of control of interacting quantum systems, recently Floquet engineering is widely used to study quantum phases of matter; e.g., simulate desirable models or construct new quasi-equilibrate phases. Its full power has not been unraveled yet.

1.2 Primary: Magnus expansion

Now we explain the basics of Floquet system, which starts from a time-periodic Hamiltonian $H(t)$ acting on a finite-dimensional Hilbert space, $\mathcal{H}$. We want to obtain its evolution operator $U(t)$, which formally is

$$U(t) = \mathcal{T} \exp\left(-i \int_0^t H(\tau) d\tau\right), \tag{1}$$

for time-ordering $\mathcal{T}$, which stands for the fact $H(t)$ and $H(t')$ do not commute in general. According to Floquet theorem, this can be further expressed as

$$U(t) = P(t)e^{-itF} \tag{2}$$

for F as an effective Hamiltonian, usually called Floquet Hamiltonian, and a unitary $P(t) = P(t + T)$. The term $P(t)$ is usually called "micromotion" since if we take a coarse-grained snapshots at stroboscopic times $t = nT$ the system is only described by the relatively "macromotion" F. Also $U(0) = P(0) = P(T) = \mathbb{1}$, $U(T) = e^{-iTF}$. However, this does not solve the problem since the forms of $P(t)$ and F are not known. It turns out this is difficult and in fact there is no concise forms of them according to Magnus expansion, which is an ansatz for $U(t)$ as an exponent of a series.

The Magnus ansatz reads

$$U(t) = \exp(\Omega(t)) = \exp\left(\sum_{k=1}^{\infty} \Omega_k(t)\right) \tag{3}$$

for the mode operators $\Omega_k(t)$. They are completely determined by

$$\dot{\Omega}(t) = \sum_{n=0}^{\infty} \frac{B_n}{n!} \mathrm{ad}_{\Omega}^n H(t) \tag{4}$$

for Bernoulli numbers B_n and adjoint operator $\mathrm{ad}_\Omega := [\Omega, \ \cdot \ ]$. The mode operators are obtained as integrals; e.g., $\Omega_1(t) = -i \int_0^t H(t_1) dt_1$.

The Magnus expansion is an ansatz since the series may not converge, just like Taylor expansion. The convergence issue has been well studied, and it usually requires a constant upper bound for $\int_0^t \|H(t_1)\| dt_1$ with a usual operator norm $\| \cdot \|$. For Floquet system with a constant period T, the convergence only needs a single period.

We can now apply Magnus expansion to the Floquet case, but still there is no simple forms for $P(t)$ and F. On the other hand, they can be nevertheless worked out for specific systems. A final crucial feature we have to mention is that there are some freedoms in choosing $P(t)$ and F. First, F is modular and there is an unphysical gauge redundancy to shift its eigenvalues, called "quasi-energies". This is like the quasi-momentum in solid-state lattice systems due to Bloch theorem. More importantly, we can use a rotating frame defined by a time-dependent unitary $G(t)$ with $U(t) = G(t)\tilde{U}(t)G^\dagger(0)$, and obtain a new Hamiltonian

$$\tilde{H}(t) = G(t)^\dagger H(t) G(t) - iG(t)^\dagger \dot{G}(t), \tag{5}$$

with $i\dot{\tilde{U}}(t) = \tilde{H}(t)\tilde{U}(t)$. We see that a term depending on $\dot{G}(t)$ can change the norm of $H(t)$, and in general this can change our point of view of the system. But, of course, finding a proper $G(t)$ is a kind of art.

1.3 More: Kapitza pendulum

How useful can Floquet engineering be? Let's see a seminal example discovered decades ago: a classical pendulum with oscillating suspension point, known as Kapitza pendulum. The drive variables are the amplitude a and the frequency ν. This system can be easily solved. The energy include the potential energy

$$V = -mg(l\cos\phi + a\cos\nu t) \tag{6}$$

with the mass m and length l of the pendulum, g as the free fall acceleration, ϕ as the angle of the rope from the vertical axis, and the kinetic energy

$$K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \tag{7}$$

with displacements $x = l\sin\phi$, $y = -l\cos\phi - a\cos\nu t$. Now from Hamilton mechanics, we can derive the equation of the dynamical variable ϕ as

$$\dot{\phi} = -\frac{1}{l}\sin\phi(g + a\nu^2\cos\nu t), \tag{8}$$

which is highly nonlinear.

The dynamics is very sensitive with $\frac{a}{l}$, and $\frac{\omega}{\nu}$ for $\omega := \sqrt{\frac{g}{l}}$. It could be chaotic, for instance. What's interesting here is that for small values of $\frac{a}{l}$ and $\frac{\omega}{\nu}$, the pendulum obtains a distinct stationary location: $\phi = \pi$ instead of $\phi = 0$. This can be solved by using a Born-Oppenheimer-type approximation to separate a slow part and a fast part of ϕ as $\phi := \phi_0 + \xi$. The slow part ϕ_0 shall be independent of ν, and it is easy to see

$$\xi = \frac{a}{l}\sin\phi_0\cos\nu t. \tag{9}$$

Then the slow part obeys

$$ml^2\ddot{\phi}_0 = -\frac{\partial V_0}{\partial\phi_0} \tag{10}$$

for an effective potential

$$V_0 = -mgl\cos\phi_0 + \frac{m}{4}(a\nu\sin\phi_0)^2. \tag{11}$$

The behavior of V_0 can be easily seen from a simple plot. It is a monotonically increasing function when $\alpha := (\frac{a}{l}\frac{\nu}{\omega})^2 \leq 2$, but when $\alpha > 2$, it develops a maximum within the range $[0, \pi]$, and the location $\phi_0 = \pi$ becomes a stable point. The maximum serves as an *energy barrier* between the two minimal at $\phi_0 = 0$ and $\phi_0 = \pi$, while the former has lower energy. Such an emergence of new stable points is called *dynamical stabilization*.

So we see that the pendulum can stay at the upper point along with fast wiggle ξ. This system can be used to encode a robust bit value! We can encode bit 0 as $\phi = 0$ and bit 1 as $\phi = \pi$, and the bit is protected by the Floquet drive against external noises. This encoding is distinct from the well-known one via magnets: bit 0 (1) as total magnetization up (down). The magnetic bit is robust against thermal noises below a critical temperature T_c, and its robustness is from the interactions among spins and the phase transition. Instead, the robustness of Kapitza pendulum bit is from the dynamical stabilization.

Furthermore, we could take a step further to ask for a stable point at arbitrary angle ϕ. It turns out this can be achieved by tuning the drive direction, θ. In the parameter space of (θ, α), there can be a minimum ϕ separated from $\phi = 0$ by an energy barrier. As a result, we can realize a sequence of stable angles

$$\phi_1 \to \phi_2 \to \phi_3 \cdots \tag{12}$$

by choosing proper drive directions. This can be done abruptly or adiabatically. We can even drive the pendulum in the full 3D space.

We wonder if the exotic Kapitza pendulum can be applied to the quantum case. There is an apparent similarity with a qubit: the qubit state can be viewed as a point on the Bloch sphere, and its evolution is just the shift of the point on the sphere. A common example of such a control process is the Rabi oscillation. For instance, a model

$$H = \omega_0 Z + \omega_1 (X \cos \omega t - Y \sin \omega t), \tag{13}$$

can prepare any state for a proper evolution time t. The Rabi oscillation has a frequency $\Omega := \sqrt{\Delta^2 + \omega_1^2}$ and amplitude $\frac{\omega_1}{\Omega}$, for detuning $\Delta := \omega - \omega_0$. We can see that at resonance $\Delta = 0$, the amplitude is maximal. What happens if $\omega \gg \omega_0$? One will find that there is no analog of dynamical stabilization (despite that designed Floquet drives can be used to perform qubit rotations). The reason is that the dimension of a qubit is too small, and we have to require a large spectrum in order to find a quantum analog of Kapitza pendulum. The strongly-correlated quantum many-body systems are such candidates.

2 Advanced topics: Floquet phases of matter

A natural task is to employ Floquet drives to manipulate quantum many-body systems, such as quantum spin chains, Hubbard models etc. The questions are what Hamiltonian, what Floquet drives, and what goals shall be chosen. Here we explain the notion of Floquet phases of matter.

A classical example of phases of matter is a magnet: it is defined by a parameterized Hamiltonian $H(\lambda)$ which have different phases separated by phase transitions in the large system limit. The models are static, in particular. For Floquet systems, this is not the case and driven systems are in highly non-equilibrium. Also, there is a *heating* puzzle: a many-body system has a large spectrum, so the system will heat up to infinite temperature in principle, and there will be no interesting physics! To avoid this, one has to carefully choose the models and the drive schemes.

People find a proper regime with local Hamiltonian and fast drives. A Hamiltonian is local if

$$H = \sum_n H_n \tag{14}$$

for H_n acting on a finite (usually constant) number of sites independent of the system size L. The terms are also bounded $\|H_n\| \leq h$ for a constant h, which sets the local energy scale. An external drive $V(t) = V(t+T)$ is said to be fast if $\omega = \frac{2\pi}{T} \gg h$, and it is also assumed to be local $V(t) = \sum_n V_n(t)$. Now the question is: can the Magnus expansion of the evolution $U(t)$ be convergent? For a constant T, the size of $\|H(t)\|T$ increases with L, so there may not be a convergence. One can choose T to be small with $1/L$, but this becomes trivial if $L \to \infty$. Instead, it is possible that the divergence occurs slowly, i.e., the system heats up slowly due to a slow absorption of energy from the external drive. For local models, this is hinted from the notable Lieb-Robinson bound

$$\|[A(t), B]\| \leq c \exp\left(-a[d_{AB} - vt]\right) \tag{15}$$

for constants c, a, v, and d_{AB} as the geometrical distance between the support of local operators A and B, and $A(t) = U^\dagger A U$ is the Heisenberg picture evolution. The bound says that a local disturbance can only spreads out with a speed v, which most likely is a constant. For Floquet drives, if $\omega \gg h$, which roughly means the system has to take many steps to absolve the energy, and this leads to the celebrated *prethermalization* theorem.

The theorem states that for a local model $H(t)$, the Floquet evolution can be written as

$$U(T) = e^{-iTF} + O(e^{-\omega/h}) \tag{16}$$

for a quasi-local Floquet Hamiltonian F. In a rotating frame $G(t) : H(t) \mapsto \tilde{H}(t)$, the model can be approximated by F up to correction $O(e^{-\omega/h})$. This means that before the prethermal time limit $\tau_* \sim e^{\omega/h}$, the system will remain in a quasi-equilibrium state defined by F. We see that the prethermal time is exponentially long of the drive ω, and this is long enough for interesting physics to occur.

The proof of the theorem is quite involved, but the idea is clear: we use Magnus expansion of the Floquet Hamiltonian and truncate the series at order $k \approx \omega/h$. The rest of the series only contribute errors in order $O(e^{-\omega/h})$ due to the locality. On the other hand, one would expect the prethermal time could be shorter for nonlocal interactions, and this is indeed the case.

Next, we explain a prethermal phases of matter: a *time crystal*, which spontaneously breaks the discrete time-translation symmetry. Namely, for a drive with period T the Floquet eigenstates in the prethermal regime have different period, NT for some fixed constant integer N. For instance, the Ising case has $N = 2$, and the order parameter will take opposite values at time T and $2T$. The broken symmetry is an *emergent* and approximate one. This can only occur for Floquet systems, with no analog for static systems. To understand it, the Floquet evolution operator $U(T)$ is approximated by

$$U(T) \approx Xe^{-iTD}, \; X^N = \mathbb{1}, \tag{17}$$

with $[D, X] = 0$ and D local. So X is a symmetry of the model D. This is a finer version of the prethermalization theorem which highlights the emergent symmetry. When it is spontaneously broken, the system is a time crystal.

A prototypical model is a half-spin system with stroboscopic Floquet Hamiltonian $H(t) = H_1$ for $t \in (0, t_1)$ and $H(t) = H_2$ for $t \in (t_1, t_2)$ with

$$H_1 = -\sum_{\langle i,j\rangle} J_{ij}\sigma_i^z\sigma_j^z - \sum_{i\alpha} h_{i,\alpha}\sigma_i^\alpha, \; H_2 = g\sum_i \sigma_i^x, \tag{18}$$

for $\vec{\sigma}$ as Pauli operators. For $T = t_1 + t_2 = 2\pi/\omega$, and $\omega \gg h$ for h as the maximal coupling strength, it can be shown that $X = \otimes_i \sigma_i^x$, and the effective model D is still two-local. This model will spontaneously break the symmetry defined by X.

Furthermore, there are other types of phases besides symmetry-breaking ones. The most notable type is the symmetry-protected phases. We will not explain this topic here; however, we need to point out that to define a Floquet phase of matter, both the micromotion $P(t)$ and F are needed. The part F is analog of static systems, but $P(t)$ is unique for Floquet systems. Sometimes it is hard to obtain a desirable static model F, which may turn out to be easier if F is generated by a Floquet scheme.

3 Most relevant theory: many-body localization

Besides the clever drive illustrated by the prethermalization theorem, we could also choose special types of models that forbid thermalization. In general, systems that do not thermalize are *non-ergodic*, and the underlying reason is mainly due to the presence of a set of explicit or emergent conserved quantities, which defines *integrability*. Systems with many-body localization (MBL) are such examples.

Consider the seminal XXZ model but with strong disorder

$$H = \sum_i J(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) + J_z \sigma_i^z \sigma_{i+1}^z + h_i \sigma_i^z, \tag{19}$$

for the local disorder h_i drawn randomly from a distribution. The model can be solved via Bethe ansatz if h_i is not random. When the disorder becomes strong, there is a MBL phase for which all eigenstates of the system do not thermalize. Namely, it violates the eigenstate thermalization hypothesis (ETH), which states that the local observable takes a thermal value

$$\lim_{t\to\infty} \langle\psi|O(t)|\psi\rangle = \mathsf{tr}(O\rho_{\text{th}}) \tag{20}$$

for $\rho_{\text{th}} \propto e^{-\beta H}$. This is a local version of the well-known ergodicity theorem. This means that for MBL the local observable will memorize its initial value forever. The integrability can be seen by unitarily transform the model as

$$H = \sum_i h_i' \tau_i^z + \sum_{ij} J_{ij} \tau_i^z \tau_{i+1}^z + \cdots \tag{21}$$

with more many-body terms ignored here. The h_i' are renormalized or dressed version of h_i. The operators τ_i^z are dressed version of local spins $\sigma_i^z = U^\dagger \tau_i^z U$, known as “lbits”, and they decay exponentially from each central site i. It is clear that the lbits are the conserved quantities, and they only interact weakly since $J_{ij} \propto e^{-|i-j|/\xi}$.

A complete theory of MBL has not been developed yet except the knowledge from specific models. Compared with thermal phases which obey the ETH, the features of MBL include:

- All eigenstates have analog properties with the ground states (GS); e.g., entanglement. If GS obeys area law, then all states do. This also means each eigenstate can be a ground state of a gapped local model.
- The Lieb-Robinson speed is very small. Namely, the Lieb-Robinson bound is linear with time, which means it takes exponentially long time for a local disturbance to spread out. This is exponentially slow that a usual gapped local system. This can also be viewed as a defining feature of *localization*.
- It can support phases of matter; i.e., it is robust against some type of perturbations. MBL phases of matter are not defined by ground states; instead, they are defined by the whole spectrum of a model. A MBL phase is separated from non-MBL (i.e. thermal) phases.

Regarding Floquet engineering, it has been shown that MBL system with Floquet drives can also support Floquet phases of matter, called Floquet MBL phases. The reason is easy to see: since the system is localized, the local Floquet drives will only have local effects, and the system can be viewed as a collection of weakly interacting local oscillators. A MBL version of time crystal has been discovered, which is more robust since the MBL is for the whole spectrum. More MBL phases have also been studied, in general. For instance, it has been shown that MBL is not compatible with non-Abelian symmetry-protected phases, but compatible with Abelian ones. The reason is that the non-Abelian symmetry demands degenerate excitations, which however is inconsistent with a set of commuting lbits.

4 Frontiers

The field of Floquet engineering is at the intersect of several fields, including quantum control, many-body physics, quantum computing, quantum transport etc. Here we continue to layout several current focuses.

One main direction we did not mention is Floquet band engineering which designs electronic band structures of electron materials. One notable example is the Floquet topological insulator. A merit compared with usual topological insulator is that the Floquet drive serves as a controllable 'switch', so that the bulk and edge modes can be controlled at will. Researchers are looking for more types of 'Floquet band' materials, and they could be used in may settings such as spintronics.

For quantum computing, an early example of Floquet scheme is the dynamical decoupling which aims to suppress decoherence of small quantum systems due to coupling with a bath. Dynamical decoupling is also widely used in many artificial systems. However, for robust (i.e. fault-tolerant) quantum computing the main framework is to use decoherence-correcting codes, which are large systems. Just like adiabatic or dissipative processes, whether Floquet drives can be used to perform robust quantum gates are generally unknown. Some recent study has constructed Floquet gates for small systems without error correction.

Ergodicity is the reason for a system to become thermally trivial, so it is of great interest to find non-ergodic system. The prethermalization, MBL, and also glassy dynamics are good examples. Recently, the many-body scars also attracts lots of interest. All of these concern the thermalization within a closed many-body quantum system, which studies local-observable properties across the whole spectrum rather than just the low-energy sector described by quantum field theories. Whether highly excited states can be described by quantum field theories is largely unexplored.

5 History, people, and story

The reason for the term "Floquet engineering" instead of "Floquet physics" is mainly this field is driven by technical development instead of fundamental physics. The theory of periodic and quasi-periodic driving was well established decades ago. Just like quantum control, the main tasks in Floquet engineering is to *design* drive schemes to achieve engineering goals instead of solving physical problems. The recent development of Floquet engineering especially in many-body system is due to the improved control technique in artificial systems such as optical lattices.

The theoretical development of Floquet phases of matter is not due to itself, instead it is due to the subjects of MBL and time crystal. The idea of time crystal came around 2010 as a spontaneous breaking of continuous-time symmetry, but this was quickly shown to be impossible for static ground states or thermal states. MBL was established around the same time which violates thermalization. So with no surprise, spontaneous breaking of discrete-time symmetry was firstly discovered in MBL systems. The prethermal version of discrete time crystal comes a bit later. A leading researcher who made crucial contributions is a young theorist Dmitry A. Abanin, whose main interest seems to be MBL instead of Floquet systems. This brief story shows that progress in physics and science often originates from the interplay among different subjects and fields, and the communication of people from different fields.

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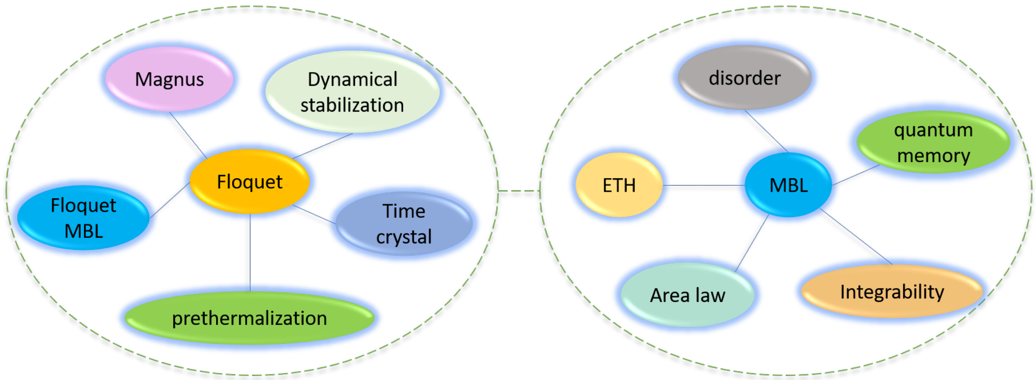
Concept map


Magnus
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