Phases of Fractons

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Definition. *A fracton model is a gapped many-body system whose excitations have restrictive mobility, and the ground-state degeneracy could be extensive.*

Keywords

Geometry; Stabilizer codes; Entanglement entropy; High-rank gauge field theory; Haah code; Self-correcting qubits; Anyon.

Abstract

This is a brief review of fracton phases of matter. We introduce fracton phases from the high-rank gauge field theory, and then we survey several simple models with fractons. Next we discuss the application of fracton phases for quantum computing, serving as logical qubits. We then discuss its relation with topological order and some of open problems in the field.

1 Minimal version

1.1 Opening

In modern condensed matter physics, topological order is a main stream. Fracton order, on the other hand, grows out of it but has distinct features.

It just has been developed for one decade, and most of researchers are very young, probably under thirty. There are some features that are believed to be characteristic, yet also some to be optional. It lacks a boost from experiment: there is no material identified as fracton matter with exotic phenomena.

Geometry, symmetry, and topology play crucial roles for fracton order, but it seems geometry is more important. One big problem in physics is how to quantize gravity, and the current understanding is based on holography. Fracton phases show some relations with gravity based on tensor gauge field theory; in this respect, it is a promising research direction.

1.2 High-rank gauge field theory

It turns out it is easier to understand fracton phases by treating them as a kind of generalized Maxwell theory. As we know, Maxwell equations describes $\vec{E}$ and $\vec{B}$ fields, and Gauss's law $\nabla \cdot \vec{E} = \rho$ specifies the gauge invariance. Charge ρ and current $\vec{J}$ are put in the theory 'by hand'; the backaction of charge or current on the field itself is not taken into account. The charge conservation says that charges cannot be simply created or destroyed, but they are free to move around.

We know that charges can form *dipoles*. How about the case when dipoles are conserved? In this case, charges cannot move freely but dipoles can. Instead of using a constrained Maxwell system, it turns out it is better to generalize it from vectors to tensors. Namely, replace $\vec{A}$ by tensor potential $A = [A_{ij}]$, $\vec{E}$ and $\vec{B}$ also be tensors $E = [E_{ij}]$, $B = [B_{ij}]$. Here, the tensors are just 3 by 3 matrices. How about charges? In Maxwell theory, charges are scalars. Here it appears there are two choices

$$\partial_i \partial_j E^{ij} = \rho, \text{ or } \partial_i E^{ij} = \rho^j, \tag{1}$$

i.e., it could be a scalar or a vector. For the first case, charges $Q = \int_V dx\rho$ of a region V and dipole moments $P^i = \int_V dx\rho x^i$ are conserved. For the second case, the vector-charge $\vec{Q} = \int_V dx\vec{\rho}$ and angular charge moment $\vec{M} = \int_V dx(\vec{\rho} \times \vec{x})$ are conserved. In total there are six of them. A vector charge can only move along the direction of its charge vector, instead of being fully static. As we will see below, the two cases describe two types of fracton phases.

The above facts make the high-rank gauge theory interesting, yet it is not fully developed. Also we shall note it is different from Yang-Mills gauge theory, which also originates from Maxwell theory.

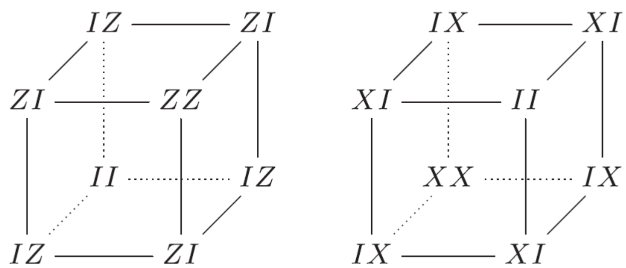

Figure 1: The two Hamiltonian terms for each cell of the Haah's model.

1.3 Toy models

Here we discuss several toy models that reveal features of fracton phases. These features are believed to be common, but it is unclear what is fundamental, i.e. as the defining features of fractons.

1.3.1 Haah code

Consider the so-called Haah's model, or Haah code. This model is defined on the 3D cubic lattice, but each site has two qubits. See figure **??**. The Hamiltonian is commuting, so it is exactly solvable. It is also a stabilizer code, which is specified by a set of stabilizers formed by products of Pauli operators. Here the stabilizers are just product of Z or X as specified. Stabilizer codes are common in quantum error correction since they describes Pauli noisy errors well, which anticommute with some stabilizers, hence can be corrected. In quantum computing, it is usually enough to only consider Pauli noisy errors.

For Haah's model, exact excitations are created by Pauli operators Z or X. Each site is shared by eight cubes, but a single-site Pauli operator only violates four Hamiltonian terms, hence four excitations (or quasiparticles). The nontrivial fact is that these four 'fractons' can only be separated by a fractal regions: this means that a single fracton cannot be moved freely and smoothly through the system. The fractal structure is a geometric effect to restrict its mobility. The ground-state degeneracy is complicated and depends on system size. It is sub-exponential with the edge size. The method to prove this is to use the stabilizer formalism. For a stabilizer code with n qubits, if there are k independent stabilizers, then the degeneracy is 2^{n-k}. So if we add more stabilizers to the Haah code, the degeneracy can be reduced.

1.3.2 X-cube model

Now, let's see the so-called X-cube model, for which some dipoles are 'anisotropic', i.e., they can move in a certain fixed direction. The model is also defined on the 3D cubic lattice but with one qubit on each edge. There are also two types of terms: the cube term is weight 12 as a product of X around a cube, and three types of vertex term each is weight 4 as a product of Z around a vertex but living on a plane. (Try to draw this!) The model is commuting and also exactly solvable. The ground-state degeneracy is exponential with the system size. Excitations by Z or X behave differently, as hinted by the different forms of stabilizer terms. A single Z operator will create four fractons, each of them cannot move freely. A single X operator will create two fractons, but each of them can move freely along a line; as such, they are called 'lineon'.

1.3.3 Classical spin model

It seems that some sort of geometric frustration is the reason for being fracton. The ground-state degeneracy seems being a result of lacking sufficient number of stabilizers. So one question is whether this is an intrinsic 3D phenomena. It turns out no. A simple model is the 4-body Ising model on a square lattice, with $-ZZZZ$ terms for each four spins around a square. It is easy to find the ground-state degeneracy is extensive. A simple one is the ground state with all spin up, and another one with all spins above a line up and below the line down, etc. These ground states are all product states, i.e. classical, but with different magnetization. The large degeneracy can also be seen as the result of subsystem symmetry, which is a line operator of X along the two directions of the lattice. The model has fractons: if we divide the system to four quadrants as two orthogonal lines, then flipping all spins in a quadrant will create a single fracton at the intersect. It cannot be moved freely. It is truly a global effect! Flipping again the spins except one row will leave two fractons. Now this pair of fractons can move along its row, so are lineons. Crucially, this model does not show any entanglement, but it has some interesting relation with classical gravity.

1.3.4 Valence-bond solids

Another interesting model is an extended valence-bond crystal on the square lattice. Given four spins sharing a square, an extended valence-bond is the superposition of $|s\rangle_{12}|s\rangle_{34}$ and $|s\rangle_{13}|s\rangle_{24}$ for $|s\rangle$ as the singlet. Now it is easy to see each spin can only share one extended bond, and there are four ground states. The interesting thing is that if there is a single unpaired spin, i.e. so-called *spinon*, then it cannot move freely with no energy cost, so it is a fracton. This is simply a *geometric frustration*. However, a pair of spinons can switch their locations with an extended bond, so they can move along one direction. This model also does not support extensive entanglement.

The last two models do not have topological order; instead, they are very classical. By definition, fracton phases do not have to be quantum or have large amount of entanglement.

Could there be one-dimensional fracton models? It seems there is no fundamental reasons to forbid this. The immobility appears similar with the idea of *confinement*, which shows up in valence-bond solids. For the $SU(3)$ AKLT model with on-site adjoint representations, which are eight-dimensional, it has two ground states. There are two types of excitations: domain wall which is $\mathbf{3} \otimes \mathbf{3}$ or its conjugate, and the 'adjointor' which is the irrep $\mathbf{8} = \mathbf{3} \otimes \bar{\mathbf{3}}$. A domain wall is a pair of $\mathbf{3}$, and this pair is free to move. However, a single irrep $\mathbf{3}$ is not free to move, i.e., it is a fracton. This is also a consequence of geometry since it can only hop to another site by inducing a dimer (i.e. singlet), which takes energy.

2 Advanced topics: self-correcting qubits

Here we discuss application of fracton order in quantum computing, namely, using them as qubits. Usually, a qubit is encoded in a logical subspace $\mathcal{C}$ as part of a total space

$$\mathcal{H} = \mathcal{C} \oplus \mathcal{S} \tag{2}$$

for $\mathcal{S}$ as the syndrome space. Ideally, our quantum information $|\psi\rangle$ is in the space $\mathcal{C}$, but noises will make leakage to the space $\mathcal{S}$. The role of error-correction is to measure some local operators, e.g., stabilizers, to identify the noises, hence we can correct them. A big task in quantum computing is to find good error-correction codes for qubits. Now a so-called self-correcting qubit is even more powerful: no error correction is needed during the computation, only when the final quantum information has to be revealed at the end. Unfortunately, people do not find any self-correcting qubits in nature yet.

A sufficient, but not necessary, condition for being self-correcting is that there is a finite temperature phase transition. This is motivated by the analog to the 2D classical Ising model, which has a critical temperature T_c and serves as the standard classical memory. We know that a generic system will thermalize at a finite temperature to a Gibbs state $e^{-\beta H}$, so how to encode information in it? The correct way is not to use Gibbs state since it cannot encode qubits, instead we will actually not allow the system to thermalize, before we finish the computation. For thermally stable phases, such as ferromagnet, the system will memorize its initial state for an exponentially long time

$$\tau \sim e^{\beta\Delta}, \tag{3}$$

which is known as the *memory time*, and here Δ is the energy barrier between any two logical gates. For 2D Ising model, $\Delta \sim L$, the system size. This is the Arrhenius law. To find the state of a ferromagnet, we just measure its magnetization.

It turns out there is one self-correcting qubit: the 4D toric code. Recall that the 2D toric code has two types of excitations: point-like magnetic anyon m and point-like electric anyon e. They are deconfined, i.e., they could move freely and the energy barrier for logical gates is a constant, so the code is not thermally stable. The 4D toric code is a full generalization of the 2D case, now the magnetic and electric excitations are loop-like, as boundaries of membrane operators. The size of loops is confined, and this makes the code thermally stable, just like the 2D Ising model. Another way to view the 4D toric code is to treat it as some kind of product of two 2D Ising models, one for the Pauli X sector, and one for Z sector.

It is now the time to see how fracton order behaves. The nature of fractons and anyons are distinct: fractons are immobile or confined, while anyons are deconfined. In other words, topological order is a kind of liquid, while fracton order is a kind of glass, as will be discussed in the next section. Let's see the 3D Haah code, which is the most well-studied one. The fractons are still point-like but now they are immobile. The operators that separate fractons apart are geometrically fractal, with the size of support as $O(\log L)$, which is basically the code distance, for linear system size L. This is also the energy barrier of the code, so we will expect that the memory time

$$\tau_{\text{Haah}} \sim L^{\beta}. \tag{4}$$

This scaling has been proven rigorously. Let's see how this is proved. The setting is the system starts from a mixed ground state $\rho(0)$, and the system is in a bath at β with Lindblad-type dissipation, and gates U are applied without error-correction in between. Within the memory time, noises cannot induce any logical error (e.g. Pauli gates). At the end, we need to compare the state $U\rho(0)U^\dagger$ with $\mathcal{D}\mathcal{G}(\rho(0))$ for $\mathcal{G} = \prod_i \mathcal{E}_i\mathcal{U}_i$, for $U = \prod_i U_i$ as a sequence of logical gates, and $\mathcal{E}_i = e^{t_i\mathcal{L}}$ as the noisy process with Lindblad superoperator $\mathcal{L}$, and $\mathcal{D}$ as the final decoding procedure, which will need to measure all stabilizers and correct errors according to the syndrome, with some classical decoder. For simplicity, we ignore the gates, and it is proved that the distance

$$d(\rho(0), \mathcal{D}(\rho(t))) \leq O(t)2^{k(L)}L^{3-c\beta}, \tag{5}$$

for constant c, $k(L)$ as the number of logical qubits. This means that for small $k(L)$, the memory time can be as big as $L^{c\beta-3}$. However, this does not grow exponentially with L and it is known that there is no critical temperature for Haah code. The code is usually called a partially or quasi self-correcting qubit. The idea to prove the bound is to first separate $\rho(t)$ into two parts: one below an energy barrier $m \sim \log L$, and the other above it. Assuming Lindblad operators are local, then it is not hard to see that the low-energy sector can be fully recovered by the decoder. Now the bound only concerns how large the high-energy sector is, which is a leakage from the code space (below the energy barrier). We will not explain the details, but note that in the bound $O(t)$ is the evolution time, i.e., the diffusion time of fractons, $2^{k(L)}$ is the logical dimension which relates to the 'size' of leakage channels, and $L^{3-c\beta}$ comes from a combinatorial fact that we have to average the effects of local Lindblad operators across the system.

3 Most relevant theory: topological order

Here we compare fracton order with topological order briefly. As we mentioned, the former is more like glass, and the later like liquid. In general, there are both immobile fractons and mobile fracton pairs in a fracton system, and the two types of excitations can have nontrivial interplay. Imagine that they have very different mobility. The fracton pairs may thermalize, then showing a volume law of entropy. While the fractons may not since each fracton can be bound to a site or a fracton pair. So, it is expected that fracton will show glassy dynamics, which is *non-ergodic* and takes very long time to reach thermal equilibrium. For Haah's model, the glassy dynamics is more apparent since there is no mobile excitations. Another perspective is many-body localization (MBL), which will prevent a system from thermalization. Here, fracton systems exhibit sort of quasi-localization without explicit randomness.

Algebraically, a topological order is defined by the set of anyons $\{a_i\}$ and their fusion rules N_{ij}^k with

$$a_i \times a_j = \sum_k N_{ij}^k a_k. \tag{6}$$

This relation actually originates from conformal field theory (CFT) with a_i known as primary fields, proved via operator-product expansion. There are also other equivalent ways to defined them, such as F and R moves, or the modular matrices S and T, with central charge c. So, what are the defining features of topological order? It seems this has not been agreed on, but there are some common ones:

1. the ground-state degeneracy depends on the topology of the manifold, e.g. genus.

2. there are edge states for manifolds with boundaries described by some CFT.

3. there are anyons in the excitations. Anyons may be of various shapes, e.g., points or loops.

Except the above, there are also other features, such as a gap for excitations, description by topological quantum field theory, as the gauged dual of some symmetry-protected topological order, or there are nontrivial braiding operations. Among these, the braiding forms the foundation for topological quantum computing. Qubits are encoded into a subspace with a fixed number of non-abelian anyons, and the braidings induce non-abelian geometric phases on the subspace, which are the gates, known as holonomy. However, non-abelian anyons have not been found in physical systems, and people are trying to simulate them.

4 Frontiers

The field of fracton phases of matter is not mature yet, although it has been a decade. There are plenty of things to work on. Below we survey some of them.

A complete field theory describing fractons has not been established. The high-rank gauge theory is only at an early stage. A recent study shows that a certain coupling between usual vector gauge field theories can also lead to fractons, as hinted by the fact that fracton models usually can be constructed by the coupling of other simpler models. For instance, the X-cube model can be viewed as a coupling among 2D surface code sheets. More connections with high-energy physics or particle physics are also needed to demonstrate the power of high-rank gauge theory.

The entanglement features of fracton models need more study. It seems both classical and quantum systems can support fractons, so the relation between entanglement and the confinement of fractons is not clear. The Haah's code shows an extensive amount of entanglement, measured by bipartite entropy, and it has been shown that there are geometrical effects besides topological ones. The interplay between geometry and entanglement is also a central issue for holography (using tensor-network approach), and indeed there are some similarity between fracton phases and gravity, both described by tensor fields. It is not clear whether topology is necessary for fracton phases or not. As fractons are more likely 3D objects, it is expected that there might be some relation with quantum gravity.

A complete *classification* of fracton phases is not known yet, which has been well established for other phases, e.g., topological phases, symmetry-protected phases. From the perspective of symmetry, topological phases spontaneously break high-form symmetries, while it seems fracton phases are often protected (i.e. unbroken) by high-form symmetries, or subsystem symmetries. There are also classification based on multipole algebra, which clearly is not fully understood/developed yet.

5 History, people, and story

Fracton models are mainly inspired by topological order. In quantum computing, people are struggling to find good qubits, and topological qubits are expected to be the best. However, as far as we know, topological order is not stable at any finite temperature. Namely, anyons induced by thermal noises are free to move, i.e. deconfined, hence they can lead to random braiding patterns and destroy the topological order. The work by J. Haah (when he was a student) inspired study of self-correcting qubits. These models are simple but strange at that time. It is only these years when the idea of fracton phases is developed that Haah's codes are understood in a general framework. This brief story tells us that usually when you are trying to find something new, and especially when you found something, you may do not understand it very well. But this is actually how science grows; science usually do not follow any plan by people with obsolete knowledge.

Another interesting fact of this field is that lots of leading researchers are very young, as graduate students or postdoc. It is hard to tell why it is so. Probably there are several factors. One is that there are no big figures, i.e. leading scientists in this field. These people may be motivated by quantum information science, and do not identify themselves as condensed matter physicists. So there is no group of people following them. Another reason is that condensed matter physicists are mainly interested in topological order, phase transition, and other topics with a notable history and connection with experiments. These topics have many connections with other topics and principles in physics, so there is no strong reason for them to jump into an unfamiliar cage.

References

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Concept map