Quantum many-body entanglement

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Definition. *Quantum many-body entanglement refers to the non-separable feature of a many-body entangled state. When expressed as a matrix-product state with a bond space, the entanglement of a state refers to the features of it in the bond space.*

Keywords

Schmidt decomposition; entanglement entropy; monotone; LOCC; matrix-product states; stabilizer states; multipartite entanglement; area law; quantum resources.

Abstract

This is a short introduction of entanglement, especially in the setting of many-body physics. We explain the basics of entanglement including its definition, measure, classification, and some notable types of entangled states based on the formalism of matrix-product states. We then introduce the area law, which is still not fully developed yet. We conclude with brief survey of quantum resource theory, some frontiers, open problems, and history of entanglement.

1 Minimal theory of entanglement

1.1 Opening

Quantum entanglement, or entanglement for short, is the most primary while also exotic features of quantum systems. The quantum systems usually contain several parts that can be individually characterized. In terms of Hilbert space, entanglement requires a tensor-product form of Hilbert spaces. Why we need Hilbert space to describe quantum systems? Well, that is the whole magic of quantum physics! One way to grasp the physical meaning of Hilbert space is to treat it as a general form of *phase space*, which is well known from classical mechanics. Entanglement refers to the nontrivial interplay between several phase spaces. A quantum state with entanglement is called an entangled state.

Why entanglement becomes vital in recent decades instead of the early age of quantum mechanics? There are many reasons. Entanglement is the property of states, instead of observable. Physicists usually only care about observable which can be measured in experiments. For instance, the exact form of quantum states for fractional quantum Hall effects is still not fully established, although observable effects such as quantized conductance, are more understood. It is due to our increasing ability to prepare and control quantum systems and use them that entanglement becomes more popular. Nowadays we are at the beginning of building *quantum computers*, the states of qubits to encode information are highly entangled states, and ultrahigh precise control of qubits are required. It is the entangled states that carry information instead of observable.

The impact of entanglement turns out to be profound. It is used for communication tasks such as secret message sharing, for metrology to increase the precision of measurement or estimation, for quantum control to enhance the ability of steering a system, for many-body systems describing exotic phases of matter such as topological order, for quantum computing to solve extremely difficult problems, and even for quantum gravity with the connection between entanglement and geometry, and many more. However, the structure of entanglement is very complicated. It needs operator algebra such as positive maps, tensor-network states and even category theory, numerical algorithms etc, and there are still many open problems which are known for a long time.

1.2 Basics

Here we survey the mathematical framework to define entanglement. Quantum states live in Hilbert spaces, which are inner product spaces, and there are two basic operations on them: tensor product $\otimes$ and direct sum $\oplus$. It turns out superposition (and coherence) is related to direct sum, while entanglement is related to tensor product. That is, for a Hilbert space $\mathcal{H} = \otimes_r \mathcal{H}_r$ as tensor product of several ones, a state $|\psi\rangle \in \mathcal{H}$ is an entangled state if it is not $\otimes_r |\psi_r\rangle$ for $|\psi_r\rangle \in \mathcal{H}_r$, which are called product states.

What does entanglement imply? On the level of states, it means all local states are mixed states. On the level of observable, it means there are correlations of some observable. It also means local observers cannot know the full information of the whole state. Also note that here "local" does not mean locality in the real space. Instead, the label r is only a mathematical counter, it does not mean a space $\mathcal{H}_r$ is carried by a physically local system. There are many situations that r refers to some abstract degree of freedom and $\mathcal{H}_r$ are supported globally.

It turns out to decide if a state is a product state is not an easy task, but this could be achieved by singular-value decomposition (SVD), also known as Schmidt decomposition. The method is to make a bipartite partition of $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$, possibly with different finite dimensions, d_a and d_b. A pure state $|\psi\rangle$ can be written as

$$|\psi\rangle = \sum_{i_a, i_b} \psi_{i_a, i_b} |i_a, i_b\rangle \tag{1}$$

with orthonormal bases $\{|i_a\rangle\}$ and $\{|i_b\rangle\}$. Now the amplitudes ψ_{i_a,i_b} altogether can be viewed as a matrix, $C = [\psi_{i_a,i_b}]$, which can be decomposed via SVD as $C = USV$ for unitary U and V and the singular-value matrix S. With simple algebra, the state $|\psi\rangle$ can be rewritten as

$$|\psi\rangle = \sum_i s_i |u_i, v_i\rangle \tag{2}$$

for singular values $s_i > 0$ with $\sum_i s_i^2 = 1$, and orthonormal bases $\{|u_i\rangle\}$ and $\{|v_i\rangle\}$ derived from U and V. This so-called *Schmidt form* highlights entanglement: if there is only one singular value, the state is a product state.

Furthermore, each of the two subsystems could have partitions on their owns. We could apply SVD again until to the finest level of partition. This actually leads to the powerful framework of matrix-product states (MPS), which usually take the form

$$|\psi\rangle = \sum_I \langle a|A^{i_1}A^{i_2}\cdots A^{i_N}|b\rangle|I\rangle \tag{3}$$

for $I = i_1, i_2, \ldots, i_N$ as the label of the N subsystems. The states $|a, b\rangle$ are boundary conditions, and the operators A^{i_n} may be of different sizes, which is roughly upper bounded by $d^{N/2}$, for d as the largest local dimension. In practice, the sizes of these A operators are taken to be the same, called the *bond dimension*. It is clear to see the bond dimension relates to the number of singular values during the sequence of SVD, which relates to entanglement of the state. We say it is a product state if the bond dimension is one.

The difficult task is for mixed states. Different from pure states above, mixed states form a convex set acting on a Hilbert space. The extremal points of the convex set are pure states. We have to deal with the relation between entanglement and convex combination, also known as *mixing*. It is widely believed that, as an axiom, mixing cannot create entanglement but may decrease the amount of entanglement. The most general mixed states without entanglement are the separable states

$$\rho = \sum_i p_i \rho_i^a \otimes \rho_i^b \cdots \tag{4}$$

with probability p_i and local states ρ_i^a etc. Separable states have classical correlations via p_i, and they form a convex set. So clearly, the set of entangled states is not convex.

What are the key differences between separable states and entangled states? There is an operational viewpoint: a separable state can be prepared via local operation and classical communication (LOCC), while an entangled state cannot. The local operations prepare the local states, and the communication is for the probability p_i. Now, given a mixed state ρ, how to know if it is entangled? This problem turns out to be very difficult, and it is known as the *separability problem* in computer science, and it is NP-hard.

Despite the difficulty, there are many methods that work very well in practice. A simple method for the bipartite case is as follows. For a state of the form $\rho = \sum_{ijkl} \rho_{ijkl}|i\rangle\langle j| \otimes |k\rangle\langle l|$, the trace norm $\|A\|_1 = \mathsf{tr}\sqrt{AA^\dagger}$ of ρ is 1. Consider any permutation π on the indices $ijkl$, and it preserves the trace norm for product states, and it decreases the trace norm for separable states.

So, if for any permutation the trace norm gets larger than 1, then the state ρ is entangled. This method appears abstract but it includes the *partial transpose* and *realignment* methods as special cases. In particular, the realignment is to permute k and j and perform an operator Schmidt decomposition, generalizing the pure-state case. The permuted state can be further rewritten as a vector $|d\rangle = \sum_{ijkl} \rho_{ijkl} |ikjl\rangle$, which is then $|d\rangle = \sum_i s_i |u_i\rangle |v_i\rangle$. But in this case the number of singular values s_i does not count entanglement; instead, the sum of them $\sum_i s_i$ has to be larger than 1. The overlap $\langle d|d\rangle$ is the purity $\mathsf{tr}\rho^2 = \sum_i s_i^2$, smaller than 1. The realignment criterion can be given a geometric meaning, which we do not pursue further here.

Treating entanglement as a kind of resource, it is also interesting to know how much entanglement it is in a state. A proper measure is usually known as an *entanglement monotone*, which does not increase under LOCC. Unfortunately, any such measure of entanglement only gives a partial answer since it induces a *partial order* on the set of entangled states. The partial order means that there are entangled states whose entanglement are not comparable. Computing a measure usually involves heavy numerical optimization, which works for small systems, though.

Another task is to classify entanglement in the sense that entangled states belonging to different classes cannot be converted to each other (via LOCC). It is hard to believe there exists such situations, but there are, and it turns out this is common. For bipartite system, all entangled states are in the same class since they can connect to the Bell states via LOCC. For tripartite system, it turns out there are two nontrivial classes: one is the GHZ class with state $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, and the other is the W class with state $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$. Their differences are the two-local reduced states, which cannot be changed by one-local operations. For four-partite system, there are almost infinite many classes according to LOCC. This is a very strong conclusion. It means that the classification via LOCC becomes trivial since it does not classify at all! One way out of this mud is to use distinct methods instead. A hint is that the LOCC-convertibility might be too restrictive. From the viewpoint of quantum error-correction codes (QECC), which use entangled states as codewords, we can use local operations of higher locality if the code distance is larger. A different method is to allow many copies of the entangled states and some "cheap" ancillary states. This setting is to imagine that the entangled state, as shared secret, is distributed among several distant users, but for some reason they want to change it to a different state. What they could do is, if they have many copies of it and local ancilla, to act on these copies and ancilla together, which is still LOCC but obviously could be more flexible than the standard setting.

1.3 More: entanglement structures

Besides entanglement measures and classifications, probably what is more important is the structures of entangled states. In the above, we have shown that any states can be written as matrix-product states, but this is just the beginning of the story. A MPS contains the following information: the set of tensors at each site $\{A^{i_n}\}$, and also boundary condition, defined by a boundary operator B. The A and B operators act on the bond space. The entanglement of a MPS is not only determined by the bond dimension, χ, but also the forms of the A and B operators. The set of A^{i_n} for each label n forms a completely positive map, which plays crucial roles for the property of the state. Here we discuss some examples to show the complexity of entanglement structures.

Stabilizer states are a wide class of states in quantum computing, including toric code, color codes, Haah code, graph states, etc. A n-qubit pure state $|\psi\rangle$ is a stabilizer state if it is "stabilized" by a set of commuting stabilizers $\{S_i\}$ with $S_i|\psi\rangle = |\psi\rangle$, and S_i is tensor product of Pauli operators. Usually, we consider local stabilizers so that there are only a constant number of neighboring Pauli operators in stabilizers. The stabilizers form a group, called the stabilizer group G_S since the product of two stabilizers is also a stabilizer. The group G_S is finite and Abelian, so it is isomorphic with $(Z_2)^n$. Although the group looks trivial, but a stabilizer state is not.

How to characterize entanglement of stabilizer states? We could use MPS but it turns out this is unnecessary. The MPS or tensor-network form of stabilizer states usually have small constant bond dimension. It is not hard to realize that its entanglement shall be simple due to the large symmetry by G_S. The simplest way is just to use $\{S_i\}$ to define its entanglement. This is a Heisenberg viewpoint which focus on observable.

Due to the simplicity of stabilizer states, an important consequence is that some dynamics of them can be classically simulated efficiently. Such dynamics is generated by the so-called Clifford operations, which maps Pauli operators to Pauli operators, so maps a stabilizer state to another stabilizer state. This is based on the Clifford hierarchy

$$\mathcal{C}^{(k)} = \{U | UPU^\dagger \in \mathcal{C}^{(k-1)}, \forall P \in \mathcal{P}_n\}, \tag{5}$$

for $\mathcal{P}_n$ as the n-qubit Pauli group, which is the product of $\mathcal{P} = \langle i\mathbb{1}, X, Z\rangle$. The Clifford group is $\mathcal{C}^{(2)}$, including the phase gate, Hadamard gate, and controlled-not gate, i.e., CNOT, which are usually easy to implement in quantum computing. With this, we can see that Clifford dynamics which preserves the set of stabilizer states can be classically simulated efficiently, just by tracking the stabilizers.

The models above defined by stabilizer states are gapped. For *gapless* models, the ground state entanglement is distinct. This could be seen from the difference between GHZ state and W state. For n qubits, the GHZ state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|00\cdots 0\rangle + |11\cdots 1\rangle) \tag{6}$$

can serve as a ground state of the Ising model, which is gapped, while the W state

$$|W\rangle = \frac{1}{\sqrt{n}}(|00\cdots 1\rangle + \cdots + |10\cdots 0\rangle), \tag{7}$$

which only has a single 1 in all branches, can serve as the ground state of a gapless spin model. In terms of MPS, the bond dimension of GHZ state is 2, while of W state is n. This leads to the entanglement entropy of W state as $\log n$, which is typical for gapless models derived first using conformal field theory. This will be explained more in the area law section below.

Another example is the *Bethe ansatz*, which is widely used in spin chains and Bose gas models. It is an ansatz of excitations whose form we do not introduce here, but it can be expressed as MPS! A Bethe state with m excitations is written as

$$|\psi_m\rangle = B_m \cdots B_2 B_1 |\Omega\rangle \tag{8}$$

for $|\Omega\rangle$ as the vacuum of a model, and B_i each creates an excitation. A key point is that these B_i are not local operators and they do not commute with each other. They could be written as matrix-product operators (MPO) which are operator analog of MPS. The bond dimension of each B_i is 2, so the bond dimension of the Bethe state is 2^m. Such a big bond dimension implies a large amount of entanglement. However, as we mentioned the entanglement is not only determined by the bond dimension, but also by the form of the tensors.

For a Bethe state, the tensors in its MPS are quite simple and constrained by the symmetry of the model, making the model exactly solvable.

Another state with an exponentially large bond dimension is the Slater *determinant state* of m free fermions. The Pauli principle requires the whole state to be antisymmetric under exchange of any two fermions. For instance, the Slater state of two fermions is a singlet $(|01\rangle - |10\rangle)/\sqrt{2}$, which appears to be an entangled state. Whether there are entanglement in identical particle system is still unresolved since the Hilbert space is not of tensor-product form. However, the global exchange symmetry has to be effectively dealt with by expressing the Slater states as entangled states. Actually, this is similar with the Bethe ansatz since each creation operator for a fermion is a MPO, and in total the bond dimension will be 2^m. The tensors in MPS are also very simple, taking similar form with Jordan-Wigner transformation, making the free fermion system exactly solvable.

These examples above show that the entanglement content of a state is not only signaled by its bond dimension in MPS, but also the tensors in it, symmetry, and correlation functions etc. We have to take all these properties into account to get a complete understanding of the *entanglement structure* of a state.

2 Advanced topics: area law

An important property of entanglement is the so-called area law (of entanglement entropy). For pure states, the bipartite entanglement can be measured by the entanglement entropy, which is the von Neumann entropy, or in general, Rényi entropies

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \log \mathsf{tr}(\rho^\alpha) \tag{9}$$

of a subsystem ρ, for $\alpha \in [0, \infty)$. The area law says for 1D (one-dimensional) quantum many-body pure states with exponentially decaying correlations functions, the entanglement entropy of a subsystem is upper bounded

$$S \leq c|\Sigma| \tag{10}$$

for a certain constant c and $|\Sigma|$ as a measure of the size of the boundary of the subsystem. It is so famous since it is the analog of black hole entropy, which scales with the surface area instead of volume of it, and this is furthermore an illustration of the *holography principle*.

How to prove the area law? It turns out to be very difficult. There are many different proofs in literature, neither one is intuitive. But it is quite easy to understand the area law: for 1D system, if correlations decay as fast as exponentially, it somehow means that the local states of two far-away subsystems are separable, i.e., no entanglement, then a fixed subsystem only correlates with neighboring regions of size determined by the correlation length, which is a constant, and this provides the upper bound $|\Sigma|$ on its entropy.

The difficulty of proving the area law is due to many facts. The correlations are local observable effects, while entropy is not a local observable, instead, it is a global quantity. The local property and global property of a quantum state are related with each other, but it is hard to tell which is more fundamental. For the area law of 1D systems, the local observable correlations determines the entropy, but it is not clear if the area law implies the exponential decay behavior. Furthermore, the role of area law is far more from clear for two and three dimensional systems. When the dimension is larger, local (point-)observable correlations tell less information of the system since there are also quasi-local observable (such as Wilson loops). In the setting of condensed-matter physics, we consider the ground states of a gapped Hamiltonian, which are described by tensor-network states or quantum field theory. There are at least five aspects that are related: exponential-decay correlations, the gap, area law, Lie-Robinson bound, efficiency of tensor-network states, but the precise logic among them is not clear at present.

For 1D systems, the implication of area law has been profound. It implies the efficiency of classical simulation, namely, the matrix-product states have bond dimensions grows at most polynomially of the system size, and this leads to the success of density-matrix renormalization group (DMRG) method. For quantum computing, it means that computation with these states cannot be universal since it can be efficiently simulated on classical computers. More powerful states have to be used to achieve universal quantum computing, and these states could be non-area law states or higher dimensional states.

3 Relevant theory: quantum resources

Entanglement is a property of a state itself, it does not depend on observable or how you make measurements on it. Of course we can use observabledependent features to describe a state, such as nonlocality, contextuality, uncertainty, and many others. These notions are generally known as quantum *resources* since they are usually aimed to be used for some tasks, including metrology, control, communication, and computation. A so-called resource theory has been developed.

In a resource theory, there are objects and operations. A set of free objects is defined and free operations are defined as those that preserve the free objects. Outside the free objects are resources, which cannot be created by free operations from free objects, and cannot be increased by free operations acting on resources. This applies to entanglement theory. The free objects are separable states, and free operations are LOCC (and a bit more), and resources are entangled states. Thermodynamics is also viewed as a resource theory. The free objects are Gibbs states, and free operations are Gibbs-preserving operations, and resources are non-Gibbs states. However, it seems resource theory has not made significant contribution to thermodynamics.

Different resources have complicated interplays. For instance, an entangled state may not show nonlocality. The nonclassical states of photons, such as squeezed states, may have negative values of its Wigner function, and this has been argued to be related with contextuality and cannot be simulated by classical means. A resource in a resource setting may not be a resource in another setting anymore. An example of this kind is the "magic" for quantum computing. For universality, there are two well-known gate set: {H,T,CNOT} and {CCZ, H}, for Hadamard gate H, controlled-not gate CNOT, Toffoli gate CCZ, and the forth-root-of-Z gate T. The H and CNOT are Clifford operations, hence can be simulated classically. The T gate promotes this to be universal for quantum computing, hence the T gate is viewed as the magic for being quantum. However, the Toffoli gate is universal for classical computation, and the H gate promotes this to be universal for quantum computing, hence the H gate should be the magic. Apparently, H generates superposition (and coherence) of states while T does not; but T generates irrational numbers while H does not. In Heisenberg picture,T generates superposition of Pauli operators. Therefore, it is hard to conclude which one is the magic. Nevertheless, resource theory brings a unified viewpoint to study different quantum resources.

4 Frontiers

Many-body entangled states are a primary subject in modern quantum theory. Theorists are trying to have a more complete understanding of them, including classification, measure, duality, observable effects, while experimentalists are trying to find them, prepare, detect, control, and use them for various applications. Here we survey several research frontiers.

The first is the interplay between entanglement and quantum field theory. To define entanglement, it requires a direct-product structure of Hilbert spaces, which in general does not exist for quantum field theories. A quantum field theory, or model, describes the dynamics of several quantum fields, which act on a space of direct-sum structure and infinite dimensional. The focus is usually on correlation functions, renormalization, phase transition etc instead of ground states, i.e., the vacuum of a model. For instance, conformal field theory can be used to describe gapless phases, including the symmetry, fusion, correlation of several primary fields. On the other hand, using tensor-network states a gapless state can be expressed as an entangled state with entanglement of $\log L$, for L as system size, and primary fields can be expressed as matrix-product operators acting on the state. So far the two approaches are consistent but there are still lots of issues to study. Furthermore, there are many other types of field theories, and their relations with entanglement also need more study.

The framework of matrix-product states (including tensor-network states) faces many problems for higher-dimensional systems. They are also called projected entangled-pair states (PEPS). The role of area law is different from that of 1D case. There are area law PEPS whose bond dimension are still too big. Also it is not sure if exponential-decay correlations implies the area law. Also to prove a 2D quantum model is gapped is far more difficult than the 1D case.

In the setting of quantum communication, which does not require a Hamiltonian for a state and the parts of a multipartite state are usually far away from each other, the role of multipartite entanglement is not clear yet, compared with bipartite entanglement. We have mentioned above that the LOCC classification might be too restrictive. Non-entangling operations beyond LOCC exist but their roles are not clear yet. Furthermore, there are many frameworks to classify multipartite entanglement, including geometric, algebraic, and operational ones, but their relations need more study. Even the simplest case is not fully understood yet: why there are two classes, the

GHZ and W, for tripartite entanglement? In the more general framework of resource theory, does there exist similar classifications?

The final subject we mention is finding more powerful quantum error-correction codes. The feature of a code is about its code distance, transversal logical gates, threshold, syndrome measurement etc. Beyond the usual stabilizer codes, more codes are developed such as holographic codes, hypergraph-product codes, fracton codes, valence-bond solid codes, codes from random circuits etc. These non-stabilizer codes have distinct entanglement structures but neither one is perfect in the sense that a single code can be universal for quantum computing using simple logical gates such as transversal ones. Besides, we also mention that an open problem in quantum computing is whether there exists thermally-stable qubit. This would require the entanglement supporting the qubit to be robust against thermal noises, namely, random excitations causing leakage from the codespace for the qubit.

5 History, people, and story

The notion of entanglement was first discussed in the famous EPR paper in 1930s right after the establish of quantum mechanics, but its importance is only realized around 1990s mainly due to the development of quantum information science. Among multidisciplinary researchers, the "Horodecki family" from Poland made many contributions, such as bound entanglement, this is recognized by their review paper in 2009 (see refs). Many-body entanglement also plays central roles in condensed-matter physics, and the origin is due to the valence-bond solid model, which was developed even earlier, in 1980s.

Another fact is that entanglement is studied by a very broad range of scientists, including mathematicians, computer(cians), physicists, chemists, and engineers etc. We can almost say no one knows or understand the whole picture of entanglement, which requires the understanding of almost all quantum physics and beyond. The so-called "2nd revolution" of quantum theory is mainly due to entanglement. Lots of progress are made not by physicists but other scientists, or by physical problems. For instance, the idea of quantum teleportation in the early 1990s was motivated by communication tasks. The whole field of quantum computing is motivated by computing problems instead of physics itself. Indeed, we believe that our understanding of entanglement will change the real world by quantum technology, and also the whole field of physics.

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Concept map