# Conformal Field Theory

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**Definition.** *A conformal field theory is a model of a class of gapless quantum systems which are defined by a few quantum fields and invariant under the conformal transformation.*

Keywords

Virasoro algebra; vertex operator algebra (VOA); primary fields; minimal models; Wess-Zumino-Witten models; spin chains; tensor networks.

Abstract

This is a brief overview of conformal field theory (CFT). We first present CFT in its minimal form based on vertex operator algebra (VOA). We then explain how CFT describes physical contents of a system, such as primary fields and their fusion rules. Furthermore, we describe two classes of models that are widely used: the minimal models and Wess-Zumino-Witten models. We conclude with a few comments on frontiers of the subject.

# 1 Minimal version of CFT

## 1.1 Opening

Half a century ago, people realized that lots of equations or systems are conformal invariant, more general than scale invariant, and this applies to critical (gapless) systems at phase transition boundaries or points. For instance, the critical point between the paramagnet and ferromagnet is a CFT. Recall that *gapless* means ground states are connected to higher-energy states in the energy spectrum of the Hamiltonian. The features of ground states for different phases differ drastically.

CFT has great impact in condensed matter physics and string theory, but not very much in other areas. The reason is that it deals with gapless systems which are very special. The other reason is that it acts on space and time, but lots of problems (such as quantum computing) do not deal with space directly.

## 1.2 Basics: Virasoro algebra

Now let us explain the basics of CFT. A CFT is just a model invariant under conformal transformation (CT). In other words, various CFT are just representations of CT, or its so-called Virasoro algebra.

It turns out spatial dimension and geometry play crucial roles for CT. Given space and time and the fields defined on it, we have to distinguish 'local' and 'global' transformation, just like local (gauge) and global symmetry. Usually local symmetry is larger than global ones, and local ones are often considered as redundancy while global ones are the real symmetry.

The global CT (gCT) is simpler than the local CT (lCT) since gCT forms a group while lCT does not. Note that as a group, gCT $\mathcal{C}$ is not compact, e.g., Lorentz group is not compact. For $\mathbb{R}^{3,1}$ with three space dimensions and one time dimension, the Lorentz group $\mathcal{L}$ is a generalized orthogonal group $O(3,1)$. With translation, it forms the Poincaré group $\mathcal{P}$. We have

$$O(3) \subset \mathcal{L} \subset \mathcal{P} \subset \mathcal{C}. \tag{1}$$

As a group, gCT $\mathcal{C}$ contains translation $P$, Lorentz map

$L$ (rotation $R$ and boost $B$), dilation (scaling) $D$, special conformal transformation (SCT) $K$. The SCT is actually a composition of inversion $I$, translation, and inversion again. Let $\rho := \begin{pmatrix} ct+z & x-iy \\ x+iy & ct-z \end{pmatrix}$, their actions are

$$L(\rho) = L\rho L,\ P(\rho) = \rho + \sigma,\ D(\rho) = \Delta\rho,\ I(\rho) = \rho^{-1}, \quad (2)$$

note generically $\rho$ is invertible (with non-zero eigenvalues $ct \pm r$).

If you are interested in the generators of them, they take the form $p_\mu = -i\partial_\mu$, $d = -ix^\mu\partial_\mu$, $l_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$, $k_\mu = -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)$. They satisfy

$$\begin{gathered} [p_\mu, p_\nu] = 0,\ [k_\mu, k_\nu] = 0,\ [d, p_\mu] = p_\mu, \\ [d, k_\mu] = k_\mu,\ [k_\mu, p_\nu] = \eta_{\mu\nu}d - il_{\mu\nu}. \end{gathered} \quad (3)$$

For CT there is also a difference between $d > 2$ and $d \leq 2$. The $d = 1$ case is not interesting for now. For $d = 2$, $x^2 = x_1^2 + x_2^2$ for $\mathbb{R}^2$ for two-dimensional classical fields, $x^2 = t^2 - x_3^2$ for $\mathbb{R}^{1,1}$ for one-dimensional quantum fields. 2D gCT is the set of Möbius map on Rienmann sphere $\mathbb{C} \cup \infty$, which contains holomorphic functions $f(z) = \alpha z + \beta$

$(z = it + x)$ as a subset. For $d > 2$, gCT is isomorphic to a generalized orthogonal group.

Contrary to gCT, the lCT does not form an infinite-dimensional group. The 2D lCT is our main interest since it is very nontrivial. For 2D cases, the classical version of lCT is the Witt algebra, and the quantum version, as central extensions with central charges, is the Virasoro algebra. The generators of lCT is

$$L_n = -z^{n+1}\partial_z. \tag{4}$$

This is from an infinitesimal shift of $z$. To see the connection with holomorphic functions for gCT, we need to take the exponent of $\sum_{n=-\infty}^{\infty} L_n v_n$. From non-singularity at $z = 0$, we see $v_n = 0$ for $n \leq -2$. From its inverse, i.e. non-singularity at $z = \infty$, we see $v_n = 0$ for $n \geq 2$. So there are only components $n = 0, \pm 1$ for gCT, which is the Möbius group $\mathcal{M}$ isomorphic to PSL(2,$\mathbb{C}$) and the restricted Lorentz group $O^+(3,1)$, which is six-dimensional. So

$$\mathcal{M} \cong PSL(2,\mathbb{C}) \cong O^+(3,1).$$

This is easy to see as a Möbius map

$$f(z) = \frac{az + b}{cz + d} \tag{6}$$

can be mapped to a matrix $[a, b; c, d]$ with nonzero determinant. A Lorentz map can be written as an invertible matrix. Their representations are different, however. Namely, a Möbius map acts on complex numbers $z$, while an invertible matrix or Lorentz map acts on the full spacetime which preserves determinant (proper distance).

From now on we will focus on the 2D lCT. The famous Virasoro algebra is

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \tag{7}$$

for central charge $c$. This is the central extension of Witt algebra which is for $c = 0$. The central charge $c$ is fixed for a particular CFT. What is the physical essence of central charge? Well it relates to many 'emergent' macroscopic quantities such as free energy. The Zamolodchikov c-theorem states that the central charge decreases monotonically under the renormalization group flow for 2D field systems. So, roughly speaking, the central charge plays similar roles with entropy.

## 1.3 Vertex operator algebra

So far we have explained the Virasoro algebra which serves as the symmetry of a gapless system. But, what is the system? Namely, what are the fields of the system? To obtain these, we follow the approach of vertex operator algebra (VOA). Usually, the total space $\mathcal{V}$ for a CFT is an infinite-dimensional Hilbert space, and there is a unique ground state (vacuum) $|\Omega\rangle \in \mathcal{V}$. The operator $L_0$, which is hermitian, is assumed to be diagonalizable and bounded from below, and the space $\mathcal{V}$ can be expressed as

$$\mathcal{V} = \bigoplus_{t=0}^{\infty} \mathcal{V}_t \tag{8}$$

for integer $t$ labeling the *weight space* $\mathcal{V}_t$, for which the weight is defined as $h \in \mathbb{R}$ with

$$L_0|h\rangle = h|h\rangle, \forall |h\rangle \in \mathcal{V}_t. \tag{9}$$

The weight is also known as conformal weight, or conformal dimension. The operator $L_0$ is often treated as the (continuous version of the) Hamiltonian of the system.

The fundamental way to introduce fields in CFT is by the *state-field correspondence*, which states that for any $|v\rangle \in \mathcal{V}$,

$$\lim_{z \to 0} Y(v, z)|\Omega\rangle = |v\rangle, \tag{10}$$

for the vertex operator

$$Y(v, z) := \sum_{m \in \mathbb{Z}} \hat{v}_m z^{-m-1}, \; z := x + it \in \mathbb{C}, \tag{11}$$

with *mode operators* $\hat{v}_m$ acting on $\mathcal{V}$. Here $x$ $(t)$ is space (time) coordinate. A vertex operator is also known as chiral operator, or chiral field. By definition, $Y(1, z) := \mathbb{1}$ is the identity operator. Physically, $Y(v, z)$ inserts a field at point $z$, which is equivalent to a boundary state $|v\rangle$. States with fixed weight are also called homogeneous states, for which the vertex operators are usually defined as

$$Y(v, z) := \sum_{m \in \mathbb{Z}} \hat{v}_m z^{-m-h}, \; |v\rangle \in V_t. \tag{12}$$

A homogeneous state $|v\rangle$ is *primary* if

$$L_m|v\rangle = 0, \; \forall m > 0,$$

and *quasi-primary* if

$$L_1|v\rangle = 0. \tag{14}$$

The Virasoro vertex operator, known as the *stress-energy tensor*, is defined as

$$L(z) := Y(\ell, z) = \sum_{m\in\mathbb{Z}} L_m z^{-m-2}, \tag{15}$$

for $|\ell\rangle$ as the Virasoro state with weight 2. We let $|\Omega\rangle \in \mathcal{V}_0$, $|\ell\rangle \in \mathcal{V}_2$. The Virasoro state (and field) is quasi-primary.

A primary field has a scaling dimension $\Delta$ from the dilation (scaling) transform $D$. If the field has spin $s$, then the holomorphic conformal dimension $h$ and its anti part are

$$h = (\Delta + s)/2, \ \bar{h} = (\Delta - s)/2. \tag{16}$$

Primary fields are invariant with respect to lCT

$$L_n\phi(z) = -z^{n+1}\partial_z\phi(z) - (n+1)hz^n\phi(z). \tag{17}$$

A primary field $\phi(z, \bar{z})$ $(\bar{z} = z^*)$ transforms as

$$\phi(z, \bar{z}) \mapsto \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}).$$

The scaling transform $D$ can be done by renormalization flow.

Similar with spins, as representation of $SU(2)$, here there are also states induced by a primary state. A *descendant* state of a primary $|\phi\rangle$ is a state generated from it by the action of any product of $L_m$ each with any order $|\phi, \vec{n}\rangle := \prod_{n_j>0} L_{-n_j}|\phi\rangle$. The span of a primary with all its descendants is called a *Verma module*, also called a conformal family, or conformal tower, denoted by $[\phi]$. It is clear that a descendant has different weight from its primary.

Primary states and descendants from different families are orthogonal. This means the total space $\mathcal{V}$ can also be written as

$$\mathcal{V} = \bigoplus_f \mathcal{V}_f \tag{19}$$

for $f$ as the label of conformal families and each $\mathcal{V}_f$ is of infinite dimension. For rational CFT, the number of conformal families is finite. In this note, we only consider CFT that are *unitary*, *diagonal*, *rational*, and being *CFT-type*. Here, unitary means the total space can be treated as a Hilbert space and also $L_{-m} = L_m^\dagger$, diagonal means each weight space (or family) is a tensor product of the meromorphic (or 'chiral') and anti-meromorphic parts, rational means the number of irreducible Verma modules is finite, and CFT-type means dim $\mathcal{V}_0 = 1$, i.e., the vacuum $|\Omega\rangle$ is unique. We usually only consider the chiral part of the whole theory.

## 1.4 Fusion rules

Primary fields could be boson, fermion, or even anyon. Primary fields do not interact, but there is a *fusion rule* on them, similar with the tensor-product of spins. Note 'fusion' does not refer to interaction. A fusion, denoted by '$\times$', between $\phi_i$ and $\phi_j$ in general takes the form

$$\phi_i \times \phi_j = \sum_k N_{ij}^k \phi_k. \tag{20}$$

The numbers $\{N_{ij}^k\}$ are fixed for a CFT and here we do not explain how to derive them. Is this an isomorphism between direct sum and product of Hilbert spaces? Do the fields have to be brought at the same location? In CFT, fusion means the generation of new fields from product of fields using operator-product expansion (OPE), so the fields do not have to be brought together. The symbol $\times$ shall not be viewed as tensor product $\otimes$ of Hilbert spaces, instead, it means there exist several fields in the state; while the symbol $+\ (\sum)$ can be treated as $\oplus$ of Hilbert spaces. So fusion means that the Hilbert space of fields on the left of fusion equation can be written as direct sum of Hilbert space of fields on the right of fusion equation. The isomorphism is by a unitary process, $U$. We can define the fusion process as $U$.

For instance, for toric code we can use a segment of Wilson loop, $C$, to create a pair of charges, $e$. This process is unitary. To fusion them, $e \times e = 1$ means it will go back to a ground state. The fusion is done by $C$ itself since it is invertible.

The OPE, also called OP algebra, can be proved rigorously. Here we show it for vertex operators, $V(\phi, z)$ for field $\phi$. The OPE takes the form

$$V(\phi, z)V(\psi, w) = V(V(\phi, z - w)\psi, w), \tag{21}$$

while the RHS can be expanded as a sum of vertex operators for fields, weighted by power-law factors. This is the underlying physics for the fusion rule. In addition, the OPE is well established in CFT, and whether it can be extended to all QFT is still an on-going research.

For non-Abelian anyons, the fusion only means that there are additional dimensions for the Hilbert space, which is the 'quantum dimension' of anyons. For Abelian anyons the quantum dimension is one, and this means that they can be created or annihilated deterministically from the vacuum. However, non-Abelian anyons cannot be created or annihilated deterministically; in order to make them unitary, additional space is required, which is the quantum dimension.

The braiding of anyons in TQFT is an additional structure different from the fusion rules. In CFT, there is no braiding. The quantum dimension supports the 'fusion space', and braiding of anyons generate unitary operations on it, which are non-Abelian holonomy. A fact is that the holonomy is equivalent to the *monodromy* of the correlation functions of primary fields, i.e., anyons. This is understood via the analytic continuation of correlation functions and the monodromy refers to the multi-valueness of them, which are the holonomy realized by braidings.

# 2 Advanced topics

The central charge $c$ plays crucial roles. Free boson and free fermion, which are equivalent, have $c = 1$ and only have Vir algebra, which can be constructed from a collection of harmonic oscillators. There is no braiding or extra symmetry. There are two classes of models with distinct values of central charges: minimal models and Wess–Zumino–Witten (WZW) models. For minimal models, $c \leq 1$, there are no extra symmetry. For Wess–Zumino–Witten (WZW) models, $c \geq 1$, and the symmetry is larger than Vir algebra.

Well known examples of minimal models include the critical Ising model with $c = 1/2$, and two nontrivial primary fields, Majorana fermion $\psi$ and Ising anyon $\sigma$, the tricritical Ising (Ising with vacant sites) with $c = 7/10$, and five nontrivial primary fields, and the $Z_3$ Potts (qutrit) model with $c = 4/5$, and five nontrivial primary fields. The $c = 1/2$ Ising is interpreted as a single Majorana fermion, and there are a family of models, e.g., WZW models, that can be viewed as $n$ Majorana fermions with $c = n/2$. This include the famous XY model with $c = 1$, as a critical region in the XXZ model. In this note, we do not derive the minimal models in details, while the basic idea is using features of Hilbert space to derive allowable values of $c$.

Next we explain the WZW model in details since it has extra structures. A WZW model is 2D CFT with a Lie-group symmetry, and the symmetry algebra is an affine Lie algebra, which is also a Kac–Moody algebra. An affine Lie algebra is an infinite-dimensional Lie algebra that is constructed in a canonical fashion out of a finite-dimensional simple Lie algebra. A WZW model starts from a simple Lie group $G$ with generators $J^a$, and then induces the loop algebra with generators $J^a_n := J^a \otimes t^n$, for $n \in \mathbb{Z}$, $t \in \mathbb{C}$, which satisfy the affine Kac-Moody algebra

$$[J^a_n, J^b_m] = if^{ab}_c J^c_{n+m} + kn\delta_{ab}\delta_{n+m,0}, \tag{22}$$

for $k$ as the level, and $k \in \mathbb{N}$ for unitary models, which implies $c \geq 1$. The number $k$ is related to the central charge $c$ by

$$c = \frac{kD}{k + v_c} \tag{23}$$

for $D$ as the dimension of the adjoint irrep of group $G$ and $v_c$ as the eigenvalue of Casimir operator in adjoint irrep. For $SU(N)$, $c = \frac{k(N^2-1)}{k+N}$.

The Virasoro operators are from the Sugawara construction

$$L_n = \frac{1}{2(k + c_g)} \sum_a \sum_{m \in \mathbb{Z}} : J^a_m J_{a,n-m} : \qquad (24)$$

for :: as ordering symbol, $c_g$ is the dual Coxeter number which is fixed for a $G$. From $[L_n, J^a_m] = -mJ^a_{n+m}$, it holds $[L_0, J^a_n] = -nJ^a_n$, which means descendants can be generated by $J^a_n$. Actually, since the symmetry is larger than the Vir algebra, states are organized into WZW conformal families with WZW primary $|v\rangle$ defined as

$$J^a_n |v\rangle = 0, \ \forall n > 0. \qquad (25)$$

There are a finite number of WZW families but an infinite number of the usual Vir families.

The current operators $J^a(z)$ are from the Laurent expansion

$$J^a(z) = \sum_n z^{-n-1} J^a_n, n \in \mathbb{Z}, \qquad (26)$$

and the OPE is

$$J^a(z)J^b(z') = \frac{k}{8\pi^2(z - z')^2} + if^{abc}\frac{J^c(z')}{2\pi(z - z')} + \cdots . \qquad (27)$$

The zero-th component $J_0^a$ satisfy the usual Lie algebra. The current operators are chiral and commute $[J_n^a, \bar{J}_m^b] = 0$. The current operators can be defined in terms of free fermion operators $L_{\alpha,n}$ and $R_{\alpha,n}$ for $n$ as species by

$$J^a(z) = \sum_n R_{\alpha,n}^\dagger(z) t_{\alpha\beta}^a R_{\beta,n}(z), \tag{28}$$

and $L$ for $z^*$. The current operators serve as lowering and raising operators to generate states of the space.

Besides the Sugawara construction, there is also an elegant form of WZW model via action. The WZW action is $S = \Gamma_0(g) + \Gamma(g)$, for 'nonlinear sigma' term

$$\Gamma_0(g) = \frac{-k}{16\pi} \int d^2x \; \mathsf{tr}(\partial_\mu g^{-1} \partial_\mu g) \tag{29}$$

and topological term

$$\Gamma(g) = \frac{ik}{24\pi} \int d\xi \int d^2x \; \epsilon^{\alpha\beta\gamma} \mathsf{tr}(\Pi_\alpha \Pi_\beta \Pi_\gamma) \tag{30}$$

for $\Pi_i = g^{-1}\partial_i g$. The integral in $\Gamma(g)$ is one-dimensional higher than the model, taking the model as the boundary, while it is a total derivative so does not depend on the bulk. This relates to the Stokes formula.

The current operators can be expressed with the $G$-valued field $g$ as

$$J(z) = -\frac{k}{2\pi} g\, \partial g^{-1},\ J(\bar{z}) = \frac{k}{2\pi} g\, \bar{\partial} g^{-1}, \tag{31}$$

and the field $g$ is called the Wess-Zumino field. For $SU(N)$, the matrix field $g$ is unitary in the fundmental irrep.

The WZW action can describe fermionic system, as such, it is also known as 'non-Abelian bosonization' method. Furthermore, it is also equivalent to Abelian bosonization by decomposing $g$ in terms of several bosonic fields, such as in sine-Gordon field theory.

In WZW model, in addition to $g$ there are also other primary fields, such as current operators. For $SU(N)_k$ WZW models, primary fields can be viewed as irreps of $SU(N)$. The $SU(2)_k$ WZW models, which are the boundary of $SU(2)_k$ Chern-Simons models, for $k > 3$ and $k \neq 4$ are proved to be universal for quantum computing. The $k = 4$ does not work since it reduces to Ising anyon case.

The $SU(N)$ WZW models can describe the critical points in $SU(N)$ spin chains, such as the dimer-VBS transition. The $SU(2)_1$ has $c = 1$, two Majorana fermions, describes

XXZ model. The $SU(2)_2$ has $c = 3/2$, three Majorana fermions, and it can describe spin-1/2 ladder which has three Majorana fermions as a triplet and another Majorana fermion as a decoupled singlet. For $SU(3)$ VBS model with adjoint irreps, the central charge is $c = 16/5$.

Fusion rules of WZW models are well developed based on representations of affine Lie algebra. For instance, for $SU(2)$ the fusion rule is just a truncated version of products between spins. There is a remarkable duality between rank and level: e.g., the model of $SU(N)_k$ is dual to $SU(k)_N$. This can be seen from the symmetry of the Young tableau. We will not discuss this in details.

# 3 Most relevant theory: topological quantum field theory

Probably the most striking fact is that a CFT can serve as the gapless edge of a topological quantum field theory (TQFT), which is gapped. We describe the duality between Chern-Simons (CS) theory and WZW theory. Chern-Simons theory can describe both non-Abelian and Abelian anyons, especially for quantum Hall states. CS model is a gauge model, i.e., it has local (gauge) symmetry (or redundancy). Usually, the gauge symmetry is a compact connected Lie group $G$, such as $U(1)$, $U(N)$, $O(N)$, and $Sp(2n)$. Given an odd-dimensional manifold $M$, such as spacetime, the connection $A$ can be defined

$$A := A^a_\mu T^a dx^\mu,\ x^\mu \in M, \tag{32}$$

for $T^a$ as generators of $G$. It is a generalized notion of 'potential' in electromagnetism. Also a 'field tensor', or curvature $F$ is

$$F := dA + A \wedge A \tag{33}$$

for exterior product $\wedge$ and differential operator $d$. In terms of components, $A \wedge A = \epsilon^{\mu\nu} A_\mu A_\nu$ for anti-symmetric tensor $\epsilon^{\mu\nu}$. The 2+1 CS action at 'level'-$k \in \mathbb{N}$ takes the form

$$S_k = \frac{k}{4\pi} \int_M \mathsf{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A). \tag{34}$$

It is so-defined since it is gauge invariant under $G$, up to $2\pi n$. The constant $n$ is proportional to the winding number $w$ of $g \in G$ in the gauge group

$$w := \frac{1}{24\pi^2} \int d^3x \epsilon^{\mu\nu\rho} \mathsf{tr}(\Pi_\mu \Pi_\nu \Pi_\rho) \tag{35}$$

for $\Pi_\mu = g^\dagger \partial_\mu g$. This is just the Wess–Zumino (WZ) term. When CS theory is defined on a manifold with boundary, then the gauge group operation will lead to the WZ term on the boundary, and this leads to the correspondence between CS theory in the bulk and WZW theory on the boundary, for the same group $G$.

We discuss the correspondence for Abelian anyon models, e.g., integer quantum Hall states, using $U(1)^N$-CS theory for $N$ species of anyons. Omitting the level $k$, the model is

$$\mathcal{L} = \frac{\epsilon^{\mu\nu\rho}}{4\pi} \sum_{I,J=1}^{N} a_\mu^I K_{IJ} \partial_\nu a_\rho^J. \tag{36}$$

The model is defined by $a^J$ and a matrix $K$, which is symmetric and has real-integer elements. When the microscopic particle is boson, all $K_{IJ}$ are even integers, while if there is an odd $K_{II}$, there will be fermions in the system. $K$ encodes the statistics of anyons.

When the model has a boundary, we know that the boundary is a CFT. Here, there are $N$ chiral boson fields $\phi^I$ for each $a^I$, which are 'compactified' $\phi^I \equiv \phi^I + 2\pi$ (i.e. periodic), since the physical observable are 'vertex operators'

$V(\phi^I) = e^{i\phi^I}$. The edge is one-dimensional so the fields are chiral: left mover $z$ and right mover $\bar{z}$. The vertex operators satisfy VOA, and the fields satisfy Kac-Moody algebra

$$[\phi_x^I, \partial_y \phi_y^J] = \pm 2\pi K_{IJ}^{-1} i\delta(x-y) \tag{37}$$

and $\pm$ for chirality. The edge can be described by a sine-Gordon field theory with terms like $\cos\phi^I$, which can induce a gap when relevant.

Such a bulk-boundary correspondence or duality is an example of *holographic principle*, the exact content of which is still elusive. It roughly states that the boundary encodes all the information of the bulk, and vice versa. However, people find that this correspondence is not one-to-one.

# 4 Frontiers

Correlation functions of exponents of primary fields $\langle \prod_n e^{i\alpha_n \phi_n} \rangle$ are the main physical observable of a CFT. However, correlation functions are especially difficult to compute, which needs operator product expansions (OPE) and vertex operator algebra (VOA), and also non-factorizable 'conformal blocks'. A CFT is called 'rational' when the $z$ part and $\bar{z}$ part factorizes and only one sector matters. Usually, we only study rational and minimal CFT. General strategy to make a CFT well defined is called 'conformal bootstrap'.

We mentioned that there is a state-field correspondence. It is not clear if this, and also the OPE, can be generalized to any quantum field theory. CFT (and related TQFT) are only a few well established quantum field theory, and there are lots of problems out there, such as renormalization, ghost fields, and especially quantum gravity. It is in general a big question whether CFT can help to make all field theories well defined.

In recent years, classical or quantum simulations of CFT using tensor-network states are of great interest. The simulation costs depend on the entanglement in CFT. It is widely open how tensor-network states look like for CFT, and how these states differ from gapped systems. On the other hand, topological qubits for quantum computing usually employ gapped phases for encoding, it is not known if CFT can be used to encode robust qubits.

## 5 History, people, and story

It took a long time for people to realize conformal invariance has to be used to generalize scale invariance in order to derive universal scaling constants. This was mainly achieved by A. Polyakov in 1970. The idea of OPE is due to K. Wilson also around 1970. The minimal models were established by Belavin, Polyakov, and Zamolodchikov in 1984 from Russian school. The WZW models were mainly established by E. Witten also around 1984. Still around the same time, J. Cardy developed CFT with boundaries, and 20 years later he developed the formula of entanglement entropy in CFT.

A mathematical line was initiatated by Frenkel, Lepowsky, and Meurman in 1988, who introduced the notion of vertex operator algebra. Notably, there are two famous Chinese mathematicians involved in this tradition: Yongchang Zhu and Yizhi Huang. However, it seems physicists and mathematicians do not talk with each other very much, and the two traditions develop quite independently.

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# Concept map