

Quantum simulation of relativistic quantum mechanics

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Abstract

We organize relativistic wave equations according to the representations of the Poincaré group and explain the relation between physical one-particle wavefunctions and the standard scalar, spinor, vector, and tensor field descriptions. For suitable massless chiral representations, the free equations can be written in a common spin–momentum form, provided that the appropriate constraints and positive-frequency conditions are included.

A second objective is to understand nonlinearity. Examples include scalar ϕ^4 and sine–Gordon theories, Dirac particles in electromagnetic backgrounds, the Maxwell–Dirac system, non-Abelian Yang–Mills theory, and Einstein’s gravity.

For spin two, we connect a field-strength wavefunction description to linearized vacuum Einstein gravity. We then explain how consistent coupling to energy and momentum, including the field’s own contribution, leads to nonlinear gravitational dynamics. Deser’s first-order construction makes this completion particularly transparent: a cubic interaction closes the theory into the Einstein–Hilbert action.

We also outline a first-quantized quantum simulation strategy for these coupled systems: matter and gauge wave functions are encoded on separate quantum registers, evolve unitarily under finite-dimensional Hamiltonians, and exchange sources and backgrounds through classical communication, so that nonlinearity arises only from the iterative coupling while each quantum step remains linear.

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1 The Poincaré group and the meaning of spin

In flat spacetime, the transformations relating inertial descriptions consist of translations and Lorentz transformations:

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu}. \quad (1)$$

They form the Poincaré group,

$$\mathcal{P} = \mathbb{R}^{1,3} \rtimes SO^{+}(1,3). \quad (2)$$

Throughout most of this article we use

$$\hbar = c = 1, \quad \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1), \quad \mathbf{p} = -i\nabla, \quad \square = \partial_t^2 - \nabla^2. \quad (3)$$

Units will occasionally be restored to clarify physical interpretations.

Quantum particle states carry unitary representations of spacetime symmetries. A particle species is characterized, in part, by the representation to which its states belong.

For a massive particle, one can choose its rest frame. The transformations that leave its rest momentum unchanged are spatial rotations. Their covering group is $SU(2)$.

An irreducible spin- s representation contains $2s + 1$ spin projections.

This classification can also be expressed through the Poincaré generators P^μ and $M^{\mu\nu}$. Define the Pauli–Lubanski vector

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}. \quad (4)$$

For a massive irreducible representation,

$$P_\mu P^\mu = m^2, \quad W_\mu W^\mu = -m^2 s(s+1). \quad (5)$$

Thus mass and spin characterize the representation independently of the choice of a particular field equation.

A massless particle has no rest frame. For ordinary finite-helicity massless representations, the physical internal label is helicity:

$$\lambda = \frac{\mathbf{J} \cdot \mathbf{p}}{|\mathbf{p}|}. \quad (6)$$

A single irreducible massless representation may have one fixed helicity. The usual photon theory contains helicities $+1$ and -1 , while linearized Einstein gravity contains $+2$ and -2 .

Internal charges, such as the color of a quark or gluon, are separate from these spacetime labels.

2 A wavefunction viewpoint and a common free-field structure

Using physical momentum eigenstates, a positive-energy relativistic one-particle wavefunction obeys

$$i\partial_t \psi_a(\mathbf{p}, t) = E_{\mathbf{p}} \psi_a(\mathbf{p}, t), \quad E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}, \quad (7)$$

where a denotes the appropriate spin, helicity, or internal label.

For a massless particle,

$$i\partial_t \psi_\lambda(\mathbf{p}, t) = |\mathbf{p}| \psi_\lambda(\mathbf{p}, t). \quad (8)$$

Let $\mathbf{S}^{(s)}$ denote dimensionless spin matrices,

$$[S_i^{(s)}, S_j^{(s)}] = i\epsilon_{ijk} S_k^{(s)}. \quad (9)$$

For the massless chiral fields considered below, a unifying quantum equation is

$$\boxed{i\partial_t \Phi_{s,\pm} = \pm \frac{1}{s} \mathbf{S}^{(s)} \cdot \mathbf{p} \Phi_{s,\pm}, \quad s > 0.} \quad (10)$$

If the angular-momentum matrices contain $\hbar$, the corresponding form is

$$i\hbar\partial_t \Phi_{s,\pm} = \pm \frac{c}{s\hbar} \hat{\mathbf{S}}^{(s)} \cdot \hat{\mathbf{p}} \Phi_{s,\pm}. \quad (11)$$

The normalization matters. For $s = 1/2$, $\mathbf{S}^{(1/2)} = \boldsymbol{\sigma}/2$, so the Hamiltonian is $\pm \boldsymbol{\sigma} \cdot \mathbf{p}$. For $s = 1$, the coefficient is one. For $s = 2$, it is one half in the tensor representation used below. Mass term can be added to describe massive particles.

Equation (10) is not physically complete yet. It must be accompanied by:

- Constraints eliminating unwanted components;
- A positive-frequency prescription;
- The correct Lorentz transformation law;
- The appropriate physical inner product.

For instance, an eigenvector of $\mathbf{S}^{(s)} \cdot \hat{\mathbf{p}}$ with eigenvalue m_s has energy $(m_s/s)|\mathbf{p}|$ in the plus branch. Constraints and frequency selection are essential for retaining only the intended physical modes.

3 Spin zero

3.1 The free scalar equation

A free scalar field satisfies the KleinGordon equation

$$(\square + m^2)\phi = 0 \quad (12)$$

with $\square = \frac{1}{c^2}\partial_t^2 - \nabla^2$. For a plane wave

$$\phi(x) = \phi_0 e^{-iEt + i\mathbf{p}\cdot\mathbf{x}}, \quad (13)$$

this gives

$$E^2 = \mathbf{p}^2 + m^2. \quad (14)$$

Restoring units,

$$\boxed{E^2 = c^2 \mathbf{p}^2 + m^2 c^4.} \quad (15)$$

In the rest frame,

$$E_0 = mc^2. \quad (16)$$

This is the relativistic energy–momentum relation. It reflects the relativistic structure built into the Klein–Gordon equation.

3.2 Dirac form of the Klein–Gordon equation

Writing $\phi = \psi_1 + \psi_2$ and demanding that the two components satisfy a first-order system

$$i\hbar \partial_t \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} a & b \\ d & c \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (17)$$

the requirement that each component obey the Klein–Gordon equation and that the eigenvalues be $E_{\pm} = \pm\sqrt{p^2 c^2 + m^2 c^4}$ fixes the parameters up to the freedom of choosing a particular solution. The choice

$$a = -c = mc^2, \quad b = d = |p|c \quad (18)$$

gives the *Dirac form*

$$\boxed{i\hbar \partial_t \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} mc^2 & |p|c \\ |p|c & -mc^2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = (mc^2 \sigma_z + |p|c \sigma_x) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.} \quad (19)$$

The same 2×2 Hamiltonian follows from the Dirac Hamiltonian $H_D = c\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta mc^2$ by writing $\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} = |p| \sigma_x \otimes I_{\text{spin}}$ and dropping the spin factor, i.e. by reducing the Lorentz group from $SO(1,3)$ to $SO(1,1)$. In this sense the Dirac form is a one-dimensional Dirac equation embedded in $3 + 1$ dimensions, with $|p|$ playing the role of the light-cone momentum.

The relation between the two equations is therefore twofold. First, every component of a Dirac spinor satisfies the Klein–Gordon equation, since $(i\hbar\gamma^\mu\partial_\mu + mc)(i\hbar\gamma^\nu\partial_\nu - mc) = -\hbar^2(\square + m^2c^2/\hbar^2)$. Second, the Klein–Gordon equation can be recast as a first-order two-component equation whose Hamiltonian is the spinless limit of the Dirac Hamiltonian. The Dirac form is thus a kinematic rewriting, not a change of statistics: in standard field theory the Klein–Gordon field is quantized as a boson and the Dirac field as a fermion.

4 Spin one

It has been found that Maxwell equations can be written in quantum form [1]. Define

$$\mathbf{E} = -\partial_t \mathbf{A} - \nabla \varphi, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (20)$$

The vacuum Maxwell equations are

$$\partial_t \mathbf{E} = \nabla \times \mathbf{B}, \quad \nabla \cdot \mathbf{E} = 0, \quad (21)$$

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E}, \quad \nabla \cdot \mathbf{B} = 0. \quad (22)$$

Introduce the Riemann–Silberstein combinations

$$\mathbf{F}_\pm = \frac{1}{\sqrt{2}}(\mathbf{E} \pm i\mathbf{B}). \quad (23)$$

Then

$$\boxed{i\partial_t \mathbf{F}_\pm = \pm \nabla \times \mathbf{F}_\pm, \quad \nabla \cdot \mathbf{F}_\pm = 0.} \quad (24)$$

The complex combination turns the pair of real evolution equations into a quantum equation.

Define

$$(S_k^{(1)})_{ij} = -i\epsilon_{kij}. \quad (25)$$

Since

$$\mathbf{S}^{(1)} \cdot \mathbf{p} = \nabla \times, \quad (26)$$

equation (24) becomes

$$i\partial_t \mathbf{F}_\pm = \pm \mathbf{S}^{(1)} \cdot \mathbf{p} \mathbf{F}_\pm. \quad (27)$$

This realizes the $s = 1$ case of (10).

For momentum along the z direction, the circular polarization vectors

$$\mathbf{e}_\pm = \frac{\mathbf{e}_x \pm i\mathbf{e}_y}{\sqrt{2}} \quad (28)$$

satisfy

$$S_z^{(1)} \mathbf{e}_\pm = \pm \mathbf{e}_\pm. \quad (29)$$

The longitudinal vector has eigenvalue zero. The constraint

$$\mathbf{p} \cdot \mathbf{F}_\pm = 0 \quad (30)$$

removes it.

For the plus branch, positive frequency selects the positive helicity; the minus branch supplies the other positive-energy helicity. Together they give the two physical photon polarizations.

For a real classical field, $\mathbf{F}_- = \mathbf{F}_+^*$. The two full complex fields are therefore not independent classical degrees of freedom.

There are gauge freedoms. The potential transformation

$$A_\mu \rightarrow A_\mu + \partial_\mu \chi \quad (31)$$

does not change $F_{\mu\nu}$.

The four components of A_μ therefore contain redundancy and constrained components. After the constraints and gauge equivalence are taken into account, two propagating polarizations remain.

Using $\mathbf{E}$ and $\mathbf{B}$ removes the potential redundancy from the variables, but retains constraints such as $\nabla \cdot \mathbf{B} = 0$. Gauge freedom and constraints are two aspects of the same physical reduction.

With a source, the Riemann–Silberstein equations become

$$i\partial_t \mathbf{F}_\pm = \pm \nabla \times \mathbf{F}_\pm - \frac{i}{\sqrt{2}} \mathbf{j}, \quad \nabla \cdot \mathbf{F}_\pm = \frac{\rho}{\sqrt{2}}. \quad (32)$$

The Maxwell equation can also be extended to massive case. A massive vector field is described by

$$\mathcal{L}_P = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_A^2 A_\mu A^\mu - j_\mu A^\mu. \quad (33)$$

Its equation is

$$\boxed{\partial_\nu F^{\nu\mu} + m_A^2 A^\mu = j^\mu.} \quad (34)$$

Taking a divergence gives

$$m_A^2 \partial_\mu A^\mu = \partial_\mu j^\mu. \quad (35)$$

For a conserved current and $m_A \neq 0$,

$$\partial_\mu A^\mu = 0. \quad (36)$$

In Proca theory this is a consequence of the equations, not a freely chosen gauge condition. In vacuum,

$$(\square + m_A^2) A^\mu = 0. \quad (37)$$

A massive vector has three physical polarizations. Its longitudinal polarization is physical.

The ordinary mass term breaks the Maxwell gauge redundancy. One can introduce a compensating scalar using the Stueckelberg formulation,

$$A_\mu - \frac{1}{m_A} \partial_\mu \sigma,$$

which is invariant under

$$A_\mu \rightarrow A_\mu + \partial_\mu \chi, \quad \sigma \rightarrow \sigma + m_A \chi.$$

This restores a redundant description, but does not change the physical count from three to two.

5 Spin one half

The Dirac equation is

$$(i\gamma^\mu \partial_\mu - m)\psi = 0, \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \quad (38)$$

In Hamiltonian form,

$$i\partial_t \psi = (\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m)\psi. \quad (39)$$

The matrices satisfy

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}, \quad \{\alpha_i, \beta\} = 0, \quad \beta^2 = 1, \quad (40)$$

and hence

$$H_D^2 = \mathbf{p}^2 + m^2.$$

The matrices $\boldsymbol{\alpha}$ are not the spin matrices. The connection to the common spin–momentum form is obtained in the massless limit, where the Dirac equation separates into two Weyl equations:

$$i\partial_t \psi_\pm = \pm \boldsymbol{\sigma} \cdot \mathbf{p} \psi_\pm. \quad (41)$$

5.1 Minimal coupling and the covariant derivative

Let q denote the electric charge, with $q = -e$ for an electron. Use

$$A_\mu = (\varphi, -\mathbf{A}), \quad D_\mu = \partial_\mu + iqA_\mu. \quad (42)$$

Then

$$(i\gamma^\mu D_\mu - m)\psi = 0 \quad (43)$$

is equivalent to

$$\boxed{i\partial_t\psi = [\boldsymbol{\alpha} \cdot (\mathbf{p} - q\mathbf{A}) + \beta m + q\varphi]\psi.} \quad (44)$$

The covariant derivative compensates for a position-dependent phase convention:

$$\psi \rightarrow e^{-iq\chi}\psi, \quad A_\mu \rightarrow A_\mu + \partial_\mu\chi. \quad (45)$$

It makes $D_\mu\psi$ transform in the same way as ψ .

In the external-field approximation, $A_\mu^{\text{ext}}(x)$ is prescribed. Equation (44) is then linear in ψ .

This approximation is useful when the electromagnetic source is sufficiently robust that the electron's backreaction and its entanglement with the source can be neglected.

A strong phase-coherent laser provides a familiar example. For appropriately chosen physical radiation modes, one may approximately describe the state as

$$|\Xi(t)\rangle \approx |\psi_e(t)\rangle \otimes |\alpha(t)\rangle, \quad A_\mu^{\text{cl}} = \langle\alpha|\hat{A}_\mu|\alpha\rangle. \quad (46)$$

Here $|\alpha(t)\rangle$ denotes coherent state. The factorization concerns a chosen physical subsystem split; electromagnetic constraint fields must still be treated consistently.

If electron-photon entanglement becomes important, the joint state has the more general structure

$$|\Xi\rangle = \sum_n c_n |e_n\rangle \otimes |\gamma_n\rangle, \quad (47)$$

and cannot be represented by one electron wavefunction in a prescribed classical field.

5.2 The Maxwell-Dirac system

Allowing the electromagnetic field to respond to the matter current gives the classical system

$$\boxed{\begin{aligned} (i\gamma^\mu D_\mu - m)\psi &= 0, \\ \partial_\nu F^{\nu\mu} &= q\bar{\psi}\gamma^\mu\psi. \end{aligned}} \quad (48)$$

Its action is

$$S = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \right]. \quad (49)$$

The coupled equations are nonlinear because

$$A_\mu \longrightarrow \psi \longrightarrow j^\mu[\psi] \longrightarrow A_\mu. \quad (50)$$

This is different from the external-field problem. It includes classical backreaction, but not an arbitrary entangled electron-photon state.

In semiclassical QED, the more general source is $\langle j^\mu \rangle$. Replacing it by a single classical spinor current requires additional assumptions. In particular, the classical self-field of a normalized single-electron wavefunction is not automatically an exact description of the electron's QED self-interaction.

The low-velocity limit is the Pauli equation. In the Dirac representation, remove the rest-energy phase:

$$\psi = e^{-imt} \begin{pmatrix} \chi \\ \zeta \end{pmatrix}, \quad \boldsymbol{\pi} = \mathbf{p} - q\mathbf{A}. \quad (51)$$

At energies small compared with m ,

$$\zeta \approx \frac{\boldsymbol{\sigma} \cdot \boldsymbol{\pi}}{2m} \chi. \quad (52)$$

Using

$$(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})^2 = \boldsymbol{\pi}^2 - q\boldsymbol{\sigma} \cdot \mathbf{B}, \quad (53)$$

one obtains

$$\boxed{i\partial_t \chi = \left[\frac{(\mathbf{p} - q\mathbf{A})^2}{2m} + q\varphi - \frac{q}{2m} \boldsymbol{\sigma} \cdot \mathbf{B} \right] \chi.} \quad (54)$$

Restoring units, the magnetic term is

$$-\frac{q\hbar}{2m} \boldsymbol{\sigma} \cdot \mathbf{B}.$$

This is the Pauli equation with the Dirac value $g = 2$.

At the next relativistic order, a static electrostatic background produces spinorbit coupling

$$H_{\text{SO}} = -\frac{q}{4m^2} \boldsymbol{\sigma} \cdot (\mathbf{E} \times \mathbf{p}) = \frac{1}{4m^2} \boldsymbol{\sigma} \cdot (\nabla V \times \mathbf{p}), \quad (55)$$

where $V = q\varphi$ and $\mathbf{E} = -\nabla\varphi$.

For a central potential,

$$H_{\text{SO}} = \frac{1}{2m^2 r} \frac{dV}{dr} \mathbf{L} \cdot \mathbf{S}. \quad (56)$$

Restoring units adds the factor $1/c^2$ to the last coefficient.

This explains the relationship between the free relativistic spin structure and atomic spin-orbit coupling: the latter is an interaction effect obtained from the low-velocity expansion, not simply the free operator $\mathbf{S} \cdot \mathbf{p}$.

6 Yang–Mills theory

6.1 The QCD action

QCD is an $SU(3)$ Yang–Mills theory coupled to quarks. A quark is a spin-1/2 field with three color components. A gluon is a spin-1 gauge field with eight adjoint color components:

$$A_\mu = A_\mu^a T^a, \quad a = 1, \dots, 8. \quad (57)$$

Color is an internal quantum number. It is independent of spatial spin or helicity. The generators satisfy

$$[T^a, T^b] = if^{abc} T^c, \quad \text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}. \quad (58)$$

To use the same sign convention as in the electromagnetic section, define

$$D_\mu q_f = (\partial_\mu + ig_s A_\mu) q_f. \quad (59)$$

Then

$$[D_\mu, D_\nu] = ig_s F_{\mu\nu}, \quad (60)$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig_s[A_\mu, A_\nu]. \quad (61)$$

In components,

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g_s f^{abc} A_\mu^b A_\nu^c. \quad (62)$$

The convention $D_\mu = \partial_\mu - ig_s A_\mu$ reverses the sign of the quadratic term and is equally common. No physical conclusion here depends on this convention.

For a matrix-valued adjoint field X ,

$$D_\mu X = \partial_\mu X + ig_s[A_\mu, X]. \quad (63)$$

The action is

$$S_{\text{QCD}} = \int d^4x \left[-\frac{1}{2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) + \sum_f \bar{q}_f (i\gamma^\mu D_\mu - m_f) q_f \right]. \quad (64)$$

Its field equations are

$$(i\gamma^\mu D_\mu - m_f) q_f = 0, \quad (65)$$

$$D_\mu F^{\mu\nu} = J^\nu, \quad (66)$$

where

$$J^{a\nu} = g_s \sum_f \bar{q}_f \gamma^\nu T^a q_f. \quad (67)$$

They are accompanied by the Bianchi identity

$$D_{[\mu} F_{\nu\rho]} = 0. \quad (68)$$

The source obeys covariant conservation,

$$D_\mu J^\mu = 0. \quad (69)$$

6.2 Maxwell-like wave equation

Define color-electric and color-magnetic fields:

$$E_i = F_{0i}, \quad B_i = -\frac{1}{2} \epsilon_{ijk} F_{jk}. \quad (70)$$

They are matrices in color space.

The Yang–Mills equations become

$$D_0 \mathbf{E} = \mathbf{D} \times \mathbf{B} - \mathbf{J}, \quad \mathbf{D} \cdot \mathbf{E} = \rho, \quad (71)$$

$$D_0 \mathbf{B} = -\mathbf{D} \times \mathbf{E}, \quad \mathbf{D} \cdot \mathbf{B} = 0, \quad (72)$$

where $\rho = J^0$ and

$$(\mathbf{D} \times \mathbf{X})_i = \epsilon_{ijk} D_j X_k. \quad (73)$$

Now define

$$\mathcal{F}_\pm = \frac{1}{\sqrt{2}} (\mathbf{E} \pm i\mathbf{B}). \quad (74)$$

Combining the two evolution equations gives

$$\boxed{iD_0 \mathcal{F}_\pm = \pm \mathbf{D} \times \mathcal{F}_\pm - \frac{i}{\sqrt{2}} \mathbf{J}, \quad \mathbf{D} \cdot \mathcal{F}_\pm = \frac{\rho}{\sqrt{2}}.} \quad (75)$$

Equivalently, with $\mathbf{\Pi} = -i\mathbf{D}$,

$$iD_0\mathcal{F}_\pm = \pm\mathbf{S}^{(1)} \cdot \mathbf{\Pi}\mathcal{F}_\pm - \frac{i}{\sqrt{2}}\mathbf{J}. \quad (76)$$

This is the non-Abelian counterpart of the Riemann–Silberstein form.

The equation is nonlinear. Equation (75) resembles a wavefunction equation, but its coefficients are not independent background numbers.

The covariant derivative contains A_μ , and the field strength itself contains $A_\mu A_\nu$. Even with $J^\mu = 0$, different color components interact.

Schematically,

$$F^2 \sim (\partial A)^2 + g_s(\partial A)AA + g_s^2 A^4. \quad (77)$$

The last two terms generate three-gluon and four-gluon vertices.

Around a vanishing background, neglecting nonlinear terms and sources gives eight free Maxwell-type systems:

$$i\partial_t\mathcal{F}_\pm^a = \pm\nabla \times \mathcal{F}_\pm^a, \quad \nabla \cdot \mathcal{F}_\pm^a = 0. \quad (78)$$

A perturbative one-gluon amplitude can therefore be labeled by

$$\psi_{a,\lambda}(\mathbf{p}), \quad a = 1, \dots, 8, \quad \lambda = \pm 1. \quad (79)$$

This is a free or perturbative particle description. It is not a description of an isolated asymptotic gluon in the confining vacuum.

Unlike electromagnetic field strengths, non-Abelian field strengths transform as

$$F_{\mu\nu} \rightarrow UF_{\mu\nu}U^{-1}.$$

Individual color components are not gauge-invariant observables.

6.3 QCD feature

Asymptotic freedom and color confinement are notable QCD phenomena. Asymptotic freedom means that sufficiently short-distance quark and gluon processes admit a weak-coupling expansion. Quantum corrections make the QCD coupling g_s depend on scale. For $SU(3)$ with n_f active quark flavors,

$$\mu \frac{dg_s}{d\mu} = -\frac{g_s^3}{16\pi^2} \left(11 - \frac{2}{3}n_f \right) + O(g_s^5). \quad (80)$$

For the quark content of QCD, the coefficient in parentheses is positive. Consequently, the coupling becomes smaller at higher momentum.

The gluonic contribution has the opposite effect from the screening due to quark–antiquark fluctuations. The non-Abelian gauge sector is essential to this behavior.

At long distances, perturbation theory fails. The observed asymptotic states are color-singlet hadrons, not isolated quarks and gluons.

In pure gauge theory, the static interaction between suitable external color sources develops a flux-tube structure. With dynamical quarks, the tube can break by producing quark–antiquark pairs, leading to color-singlet hadrons.

Confinement is a nonperturbative property supported by experiment and lattice calculations. It does not follow as a complete mathematical proof from the one-loop running coupling alone.

7 Spin two

7.1 A symmetric tensor version of the wavefunction

Consider a complex symmetric traceless spatial tensor $\mathcal{W}_{ij}$. It has five components under spatial rotations.

Define the spin-two matrices by their action:

$$(S_k^{(2)} X)_{ij} = -i(\epsilon_{kil} X_{lj} + \epsilon_{kjl} X_{il}). \quad (81)$$

Define a symmetric curl,

$$(\text{Curl } X)_{ij} = \frac{1}{2}(\epsilon_{ikl} \partial_k X_{lj} + \epsilon_{jkl} \partial_k X_{il}). \quad (82)$$

Then

$$\mathbf{S}^{(2)} \cdot \mathbf{p} = 2 \text{Curl}. \quad (83)$$

The free chiral equations are

$$\boxed{i\partial_t \mathcal{W}_\pm = \pm \text{Curl } \mathcal{W}_\pm = \pm \frac{1}{2} \mathbf{S}^{(2)} \cdot \mathbf{p} \mathcal{W}_\pm,} \quad (84)$$

together with

$$(\mathcal{W}_\pm)_{ij} = (\mathcal{W}_\pm)_{ji}, \quad (\mathcal{W}_\pm)_{ii} = 0, \quad \partial_i (\mathcal{W}_\pm)_{ij} = 0. \quad (85)$$

On transverse traceless tensors,

$$\text{Curl}^2 = -\nabla^2,$$

so

$$\square \mathcal{W}_\pm = 0.$$

The positive-frequency sectors carry helicities $+2$ and -2 .

To connect this construction to gravity, write

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \quad \kappa^2 = 32\pi G. \quad (86)$$

We use

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}, \quad (87)$$

and $R_{\mu\nu} = R^\rho_{\mu\rho\nu}$.

Let $C_{\mu\nu\rho\sigma}^{(1)}$ be the linearized Weyl tensor. In linearized vacuum gravity it agrees with the linearized Riemann tensor. Define

$$\mathcal{E}_{ij} = C_{0i0j}^{(1)}, \quad \mathcal{B}_{ij} = -\frac{1}{2}\epsilon_{ikl} C_{kl0j}^{(1)}. \quad (88)$$

The linearized vacuum equations and Bianchi identities imply

$$\dot{\mathcal{E}} = \text{Curl } \mathcal{B}, \quad \dot{\mathcal{B}} = -\text{Curl } \mathcal{E}, \quad (89)$$

as well as the symmetry, trace, and divergence constraints.

Thus

$$\mathcal{W}_\pm = \mathcal{E} \pm i\mathcal{B} \quad (90)$$

satisfies (84).

The spin-two field-strength equation therefore has a precise connection to linearized vacuum Einstein gravity.

The free massless spin-two action can be written as

$$S_2 = \int d^4x \left[\frac{1}{2} \partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} - \partial_\mu h^{\mu\nu} \partial^\lambda h_{\lambda\nu} + \partial_\mu h^{\mu\nu} \partial_\nu h - \frac{1}{2} \partial_\lambda h \partial^\lambda h \right], \quad (91)$$

where $h = \eta^{\mu\nu} h_{\mu\nu}$.

It is invariant, up to a boundary term, under

$$\delta_0 h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \quad (92)$$

Define

$$G_{\mu\nu}^L[h] = \frac{1}{2} \left[\partial_\rho \partial_\mu h^\rho{}_\nu + \partial_\rho \partial_\nu h^\rho{}_\mu - \square h_{\mu\nu} - \partial_\mu \partial_\nu h - \eta_{\mu\nu} (\partial_\rho \partial_\sigma h^{\rho\sigma} - \square h) \right]. \quad (93)$$

The field equation is

$$G_{\mu\nu}^L[h] = 0. \quad (94)$$

The operator satisfies the off-shell identity

$$\boxed{\partial^\mu G_{\mu\nu}^L \equiv 0.} \quad (95)$$

With

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h,$$

the harmonic gauge condition

$$\partial^\mu \bar{h}_{\mu\nu} = 0$$

gives

$$G_{\mu\nu}^L = -\frac{1}{2} \square \bar{h}_{\mu\nu}. \quad (96)$$

Thus the vacuum equation becomes

$$\square \bar{h}_{\mu\nu} = 0.$$

For vacuum radiation, transverse traceless gauge makes the two physical polarizations explicit. For propagation along z ,

$$h_{ij}^{\text{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (97)$$

The circular combinations are the helicity basis. They are not extra degrees of freedom beyond the two linear polarizations.

For our metric and curvature conventions, define

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}. \quad (98)$$

The first-order matter coupling is

$$S_{\text{int}}^{(1)} = -\frac{\kappa}{2} \int d^4x h_{\mu\nu} T^{\mu\nu}. \quad (99)$$

With the normalization in (91), the linear sourced equation is

$$G_{\mu\nu}^L[h] = \frac{\kappa}{4} T_{\mu\nu}. \quad (100)$$

Taking a divergence and using (95) requires

$$\partial^\mu T_{\mu\nu} = 0.$$

The stress tensor of free matter obeys this condition. Once matter exchanges energy and momentum with $h_{\mu\nu}$, its original free stress tensor no longer obeys the same separate ordinary conservation equation.

A consistent interacting theory must account for the energy and momentum of the field and its interactions.

7.2 Bootstrap to nonlinear Einstein gravity

Deser developed the seminal bootstrap approach to derive Einstein's field equation from the linear limit [2].

The quadratic action produces a stress tensor quadratic in the field. Coupling the field to this stress tensor produces cubic terms. Those interactions themselves contribute to the stress tensor, requiring further terms.

Schematically,

$$S = S_2 + \kappa S_3 + \kappa^2 S_4 + \cdots. \quad (101)$$

The gauge transformation is also deformed:

$$\delta = \delta_0 + \kappa \delta_1 + \cdots. \quad (102)$$

Consistency requires

$$\delta_0 S_2 = 0, \quad (103)$$

$$\delta_0 S_3 + \delta_1 S_2 = 0, \quad (104)$$

$$\delta_0 S_4 + \delta_1 S_3 + \delta_2 S_2 = 0. \quad (105)$$

These conditions ensure that adding interactions does not activate the components that the free gauge structure removed from the physical spectrum.

The free theory is consistent on its own. The need for nonlinear completion arises when one demands the relevant local, dynamical couplings.

Deser simplifies the iteration by introducing two independent fields:

- a symmetric contravariant density perturbation $H^{\mu\nu}$;
- a connection-like auxiliary field $C_{\mu\nu}^\alpha = C_{\nu\mu}^\alpha$.

At the free level,

$$H^{\mu\nu} = -h^{\mu\nu} + \frac{1}{2}\eta^{\mu\nu}h. \quad (106)$$

Thus H is not the same variable as the ordinary metric perturbation h .

Define

$$K_{\mu\nu}(C) = \partial_\alpha C_{\mu\nu}^\alpha - \partial_\nu C_{\mu\alpha}^\alpha, \quad (107)$$

$$Q_{\mu\nu}(C) = C_{\mu\nu}^\alpha C_{\alpha\beta}^\beta - C_{\beta\mu}^\alpha C_{\alpha\nu}^\beta. \quad (108)$$

A first-order version of the free action is

$$S_2^{\text{first}} = -2 \int d^4x [H^{\mu\nu} K_{\mu\nu}(C) + \eta^{\mu\nu} Q_{\mu\nu}(C)]. \quad (109)$$

Eliminating C reproduces the free spin-two theory.

The self-coupling is supplied by

$$\boxed{S_3^{\text{first}} = -2\kappa \int d^4x H^{\mu\nu} Q_{\mu\nu}(C).} \quad (110)$$

The complete expression is therefore

$$S = -2 \int d^4x [H^{\mu\nu} K_{\mu\nu} + (\eta^{\mu\nu} + \kappa H^{\mu\nu}) Q_{\mu\nu}]. \quad (111)$$

Now define

$$\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + \kappa H^{\mu\nu}, \quad \Gamma_{\mu\nu}^\alpha = \kappa C_{\mu\nu}^\alpha. \quad (112)$$

Then

$$R_{\mu\nu}(\Gamma) = \kappa K_{\mu\nu}(C) + \kappa^2 Q_{\mu\nu}(C). \quad (113)$$

Equation (111) becomes, up to a boundary term,

$$\boxed{S = -\frac{2}{\kappa^2} \int d^4x \mathfrak{g}^{\mu\nu} R_{\mu\nu}(\Gamma).} \quad (114)$$

In the nondegenerate Lorentzian sector, identify

$$\mathfrak{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu}. \quad (115)$$

The action is the Palatini form of Einstein gravity.

Varying the symmetric connection gives metric compatibility, which can be written as

$$\nabla_\alpha \mathfrak{g}^{\mu\nu} = 0. \quad (116)$$

It fixes the connection to the Levi-Civita connection:

$$\Gamma_{\mu\nu}^\alpha = \frac{1}{2} g^{\alpha\rho} (\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}). \quad (117)$$

Eliminating the auxiliary connection yields

$$S_{\text{EH}} = -\frac{2}{\kappa^2} \int d^4x \sqrt{-g} R = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} R. \quad (118)$$

The minus sign follows from our combined metric, curvature, and stress-tensor conventions. Changing the curvature convention changes this displayed sign.

Consistent universal coupling organizes the matter action as $S_m[g, \text{matter}]$. After eliminating the auxiliary connection, spinor matter can be included using a tetrad and its torsion-free spin connection.

Varying

$$S_{\text{EH}} + S_m \quad (119)$$

gives

$$-\frac{2}{\kappa^2} G_{\mu\nu} + \frac{1}{2} T_{\mu\nu} = 0. \quad (120)$$

Hence

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}.} \quad (121)$$

All matter responds to the same effective metric. This is the point at which the field-theory construction acquires its common geometric interpretation.

8 Quantum simulation of wave equation and nonlinearity

Consider a multicomponent field amplitude

$$F(t) \in \mathcal{H},$$

where $\mathcal{H}$ is an appropriate function space. After spatial discretization, $\mathcal{H}$ can simply be a finite-dimensional vector space.

Suppose that F obeys

$$\boxed{i\hbar\partial_t F(t) = H(t)F(t) + J(t).} \quad (122)$$

Here:

- $H(t)$ is a prescribed linear operator;
- $J(t)$ is a prescribed source vector;
- neither $H(t)$ nor $J(t)$ depends on the unknown $F(t)$.

The source J has the dimensions of energy times field amplitude. Its precise relation to a physical current depends on the theory.

Equation (122) is linear but inhomogeneous. Its solutions do not form a vector space: adding two solutions doubles the source. However, the difference between two solutions driven by the same source satisfies a homogeneous equation. This elementary observation is the basis of the construction.

Choose a particular solution F_p satisfying

$$i\hbar\partial_t F_p(t) = H(t)F_p(t) + J(t). \quad (123)$$

Define a shifted field

$$\boxed{\tilde{F}(t) = F(t) - F_p(t).} \quad (124)$$

Subtracting (123) from (122) gives

$$\boxed{i\hbar\partial_t \tilde{F}(t) = H(t)\tilde{F}(t).} \quad (125)$$

The initial condition becomes

$$\tilde{F}(t_0) = F(t_0) - F_p(t_0). \quad (126)$$

Thus the full solution is decomposed as

$$\boxed{F(t) = F_p(t) + \tilde{F}(t),} \quad (127)$$

where:

- F_p contains the response to the external source;
- $\tilde{F}$ propagates according to the source-free evolution equation.

The transformation is exact. It does not require weak sources or a perturbative expansion.

What is absorbed is the accumulated response to J , not merely the instantaneous vector $J(t)$. A replacement such as $\tilde{F} = F + J$ is therefore not generally correct.

The decomposition is not unique. If χ satisfies the homogeneous equation, then

$$F'_p = F_p + \chi \quad (128)$$

is another particular solution. The corresponding remainder is

$$\tilde{F}' = \tilde{F} - \chi, \quad (129)$$

so the total field is unchanged.

A particular solution becomes specified once appropriate initial, boundary, or retarded-response conditions are chosen.

This freedom is a freedom of decomposition. It should not be confused with a gauge transformation: the source response can contain physically measurable field strength and energy.

We now apply the method above to the wave equations with external sources.

Introduce a prescribed source in the same representation:

$$\boxed{i\hbar\partial_t F_{s,\zeta} = H_{s,\zeta} F_{s,\zeta} + J_{s,\zeta}.} \quad (130)$$

Let $F_{s,\zeta}^{\text{p}}$ solve the corresponding driven equation, including the required constraints. Define

$$\boxed{\tilde{F}_{s,\zeta} = F_{s,\zeta} - F_{s,\zeta}^{\text{p}}.} \quad (131)$$

The remainder obeys

$$\boxed{i\hbar\partial_t \tilde{F}_{s,\zeta} = \zeta \frac{c}{s\hbar} \hat{\mathbf{S}}^{(s)} \cdot \hat{\mathbf{p}} \tilde{F}_{s,\zeta}.} \quad (132)$$

Thus the same spin-momentum generator governs the source-free remainder for every spin sector in this representation.

The particular solution is

$$\begin{aligned} F_{s,\zeta}^{\text{p}}(t) &= U_{s,\zeta}(t, t_0) F_{s,\zeta}^{\text{p}}(t_0) \\ &\quad - \frac{i}{\hbar} \int_{t_0}^t dt' U_{s,\zeta}(t, t') J_{s,\zeta}(t'), \end{aligned} \quad (133)$$

where

$$U_{s,\zeta}(t, t') = \exp \left[-\frac{i}{\hbar} (t - t') H_{s,\zeta} \right] \quad (134)$$

for a time-independent free generator.

For a translation-invariant problem, the generator is a finite matrix at each momentum:

$$H_{s,\zeta}(\mathbf{p}) = \zeta \frac{c}{s\hbar} \hat{\mathbf{S}}^{(s)} \cdot \mathbf{p}. \quad (135)$$

The source response can be computed mode by mode:

$$\begin{aligned} F_{s,\zeta}^{\text{p}}(\mathbf{p}, t) &= e^{-iH_{s,\zeta}(\mathbf{p})(t-t_0)/\hbar} F_{s,\zeta}^{\text{p}}(\mathbf{p}, t_0) \\ &\quad - \frac{i}{\hbar} \int_{t_0}^t dt' e^{-iH_{s,\zeta}(\mathbf{p})(t-t')/\hbar} J_{s,\zeta}(\mathbf{p}, t'). \end{aligned} \quad (136)$$

This is particularly useful for Fourier-based simulation: spatial propagation becomes multiplication by a small spin-dependent matrix at each momentum.

After the source response has been subtracted, each physical positive-energy helicity amplitude evolves as

$$\tilde{f}_\lambda(\mathbf{p}, t) = e^{-ic|\mathbf{p}|(t-t_0)/\hbar} \tilde{f}_\lambda(\mathbf{p}, t_0). \quad (137)$$

The reduction to this expression assumes that the physical constraints and frequency conditions have already been imposed.

9 Quantum simulation for the first-quantized Dirac–Maxwell system

We consider the Dirac–Maxwell system in a *first-quantized* framework. The electron is described by a four-component wave function, and the electromagnetic field by a Riemann–Silberstein wave function. Both are c-number objects encoded as quantum-state amplitudes. No field operators and no second quantization are used. The two sectors are simulated on separate quantum registers and coupled through classical communication.

9.1 Electron register

Let N_s be the number of spatial lattice sites. At each site n , the electron wave function is a four-component spinor

$$\psi_n(t) = (\psi_{n,1}(t), \psi_{n,2}(t), \psi_{n,3}(t), \psi_{n,4}(t))^T \in \mathbb{C}^4. \quad (138)$$

The full first-quantized wave function is

$$\Psi(t) = \bigoplus_{n=1}^{N_s} \psi_n(t) \in \mathbb{C}^{4N_s}, \quad \sum_{n=1}^{N_s} \sum_{a=1}^4 |\psi_{n,a}(t)|^2 = 1. \quad (139)$$

It is encoded on register E as

$$|\Psi_e(t)\rangle = \sum_{n=1}^{N_s} \sum_{a=1}^4 \psi_{n,a}(t) |n\rangle \otimes |a\rangle. \quad (140)$$

Here $|n\rangle$ labels the lattice site and $|a\rangle$ the four spinor components. The amplitudes $\psi_{n,a}(t)$ are the complex amplitudes of the quantum state; they are not operators.

Given a classical background $A_\mu^{(k)}(x)$, the Dirac Hamiltonian is the Hermitian matrix acting on $\mathbb{C}^{4N_s}$,

$$H_D[A^{(k)}] = \sum_n \beta m c^2 \otimes |n\rangle\langle n| + \sum_{\langle n,m \rangle} (-i\hbar c \boldsymbol{\alpha} \cdot \hat{\mathbf{e}}_{nm}) e^{ieA_{nm}^{(k)}/\hbar} \otimes |n\rangle\langle m| + \sum_n eA_{0,n}^{(k)} I_4 \otimes |n\rangle\langle n|, \quad (141)$$

where $\beta, \boldsymbol{\alpha}$ are the 4×4 Dirac matrices and $A_{nm}^{(k)}$ are c-number link variables. The wave function evolves as

$$i\hbar \partial_t \Psi(t) = H_D[A^{(k)}] \Psi(t), \quad (142)$$

implemented on register E by Trotterized unitary gates

$$U_D^{(k)} = e^{-iH_D[A^{(k)}]\Delta t/\hbar}. \quad (143)$$

The mass term, lattice hopping, Peierls phases, and scalar potential are realized by single-qubit phase gates and controlled exchange gates.

The electron charge and current densities are c-number bilinears of the wave function,

$$\rho_n(t) = -e \sum_{a=1}^4 |\psi_{n,a}(t)|^2, \quad \mathbf{j}_n(t) = -ec \sum_{a,b=1}^4 \psi_{n,a}^*(t) \boldsymbol{\alpha}_{ab} \psi_{n,b}(t). \quad (144)$$

In the quantum simulation they are obtained as expectation values of the single-particle operators

$$\hat{\rho}_n = -e I_4 \otimes |n\rangle\langle n|, \quad \hat{\mathbf{j}}_n = -ec \boldsymbol{\alpha} \otimes |n\rangle\langle n|, \quad (145)$$

which are finite-dimensional matrices, not field operators.

9.2 Photon register

The electromagnetic sector is described by the Riemann–Silberstein wave function

$$\mathbf{F}^\pm(\mathbf{x}, t) = \sqrt{\frac{\epsilon_0}{2}}(\mathbf{E}(\mathbf{x}, t) \pm ic\mathbf{B}(\mathbf{x}, t)). \quad (146)$$

After the transversality condition $\nabla \cdot \mathbf{F}^\pm = 0$, only two physical polarizations remain. At each lattice site we encode these two polarizations on register P as

$$|\Psi_\gamma(t)\rangle = \sum_{n=1}^{N_s} \sum_{\lambda=\pm} F_{n,\lambda}(t) |n\rangle \otimes |\lambda\rangle, \quad (147)$$

where $F_{n,\lambda}(t)$ are c-number amplitudes.

The free Hamiltonian is the finite-dimensional matrix

$$H_\gamma = \pm c \left(\hat{\mathbf{S}}^{(1)} \cdot \hat{\mathbf{p}} \right), \quad (148)$$

where $\hat{\mathbf{S}}^{(1)}$ are the 3×3 spin-1 matrices and $\hat{\mathbf{p}}$ is the lattice momentum operator. In the basis $|n\rangle \otimes |\lambda\rangle$, its matrix elements are

$$\langle n, \lambda | H_\gamma | m, \lambda' \rangle = \pm c \langle n | \hat{p}_i | m \rangle (\hat{S}_i^{(1)})_{\lambda\lambda'}. \quad (149)$$

After restricting to the two transverse polarizations, H_γ reduces to a Hermitian $2N_s \times 2N_s$ matrix acting on the photon wave function. It does not contain $\mathbf{F}^\pm$.

The photon wave function evolves as

$$i\hbar \partial_t \mathbf{F}^\pm = H_\gamma \mathbf{F}^\pm, \quad (150)$$

implemented by Trotterized unitary gates $e^{-iH_\gamma \Delta t/\hbar}$ on register P . The lattice hopping and polarization mixing are realized by controlled exchange gates and single-qubit phase gates; the transversality condition is maintained by encoding only the two transverse polarizations and projecting the initial state.

For a given classical source $(\rho^{(k)}, \mathbf{j}^{(k)})$, the inhomogeneous photon equation is

$$i\hbar \partial_t \mathbf{F}^\pm = H_\gamma \mathbf{F}^\pm - \frac{i\hbar}{\sqrt{2\epsilon_0}} \mathbf{j}^{(k)}. \quad (151)$$

We decompose it into a particular solution and a homogeneous remainder,

$$\mathbf{F}^\pm = \mathbf{F}_p^\pm + \tilde{\mathbf{F}}^\pm, \quad (152)$$

where

$$i\hbar \partial_t \mathbf{F}_p^\pm = H_\gamma \mathbf{F}_p^\pm - \frac{i\hbar}{\sqrt{2\epsilon_0}} \mathbf{j}^{(k)}, \quad i\hbar \partial_t \tilde{\mathbf{F}}^\pm = H_\gamma \tilde{\mathbf{F}}^\pm. \quad (153)$$

The homogeneous part is evolved unitarily on register P . The source response $\mathbf{F}_p^\pm$ is a known c-number function of the classical current and is added by a controlled displacement

$$|\mathbf{F}^\pm\rangle \mapsto |\mathbf{F}^\pm + \mathbf{F}_p^\pm\rangle \quad (154)$$

using an auxiliary qubit and controlled addition gates. Thus the photon step is also fully quantum; no classical post-processing of the photon state is required.

From the photon wave function one reconstructs the classical fields

$$\mathbf{E} = \frac{1}{\sqrt{2\epsilon_0}}(\mathbf{F}^+ + \mathbf{F}^-), \quad \mathbf{B} = \frac{1}{ic\sqrt{2\epsilon_0}}(\mathbf{F}^+ - \mathbf{F}^-), \quad (155)$$

and hence the classical potential $A_\mu^{(k+1)}$ used in the next electron step.

9.3 Classical communication and iterative protocol

The two registers are not coupled into a single Hilbert space. They are evolved separately and coupled through classical communication of expectation values. At step k :

1. Evolve the electron state on E with $U_D^{(k)} = e^{-iH_D[A^{(k)}]\Delta t/\hbar}$.
2. Measure the single-particle densities on E to obtain $\rho^{(k+1)}$ and $\mathbf{j}^{(k+1)}$.
3. Evolve the photon state on P with the free unitary $e^{-iH_\gamma\Delta t/\hbar}$ and apply the controlled source displacement $\mathbf{F}_p^{\pm, (k+1)}$.
4. Measure the photon wave function on P to obtain $\mathbf{E}^{(k+1)}$ and $\mathbf{B}^{(k+1)}$, and reconstruct $A_\mu^{(k+1)}$.
5. Repeat for $k = 0, \dots, K-1$.

The transversality condition is maintained by encoding only the two transverse polarizations and projecting the initial state. The Gauss-law constraint is enforced by a penalty term in the Hamiltonian or by restricting to the constrained subspace,

$$\left(\nabla \cdot \hat{\mathbf{E}}_n - \hat{\rho}_n\right) |\Psi_{\text{phys}}\rangle = 0. \quad (156)$$

9.4 Resources and limitations

The electron register requires $2N_s$ qubits (one for particle/antiparticle and one for spin) or $\lceil \log_2 N_s \rceil + 2$ qubits with binary site encoding. The photon register requires N_s qubits for two circular polarizations. Each Trotter step uses $O(N_s)$ two-qubit gates, and the total gate count scales as $O(KN_s)$ with $K = t/\Delta t$. The scheme is first-quantized: particle number is fixed to one, and electron–positron pair creation, spontaneous emission, and photon-number fluctuations are not described. The nonlinearity of the coupled system appears only through the iterative classical communication; each individual quantum evolution is linear and unitary.

First-quantized Dirac–Maxwell simulation protocol.

1. **Electron register E :** encode $\Psi \in \mathbb{C}^{4N_s}$ as $|\Psi_e\rangle = \sum_{n,a} \psi_{n,a} |n, a\rangle$. Evolve with $e^{-iH_D[A^{(k)}]\Delta t/\hbar}$.
2. **Photon register P :** encode transverse $\mathbf{F}^\pm$ as $|\Psi_\gamma\rangle = \sum_{n,\lambda} F_{n,\lambda} |n, \lambda\rangle$. Evolve with $e^{-iH_\gamma\Delta t/\hbar}$ and apply controlled displacement by the classical source response $\mathbf{F}_p^\pm$.
3. **Classical communication:** measure $\rho, \mathbf{j}$ on E ; measure $\mathbf{E}, \mathbf{B}$ on P ; reconstruct A_μ ; update the Hamiltonians.
4. **Constraints:** encode only transverse photons; enforce Gauss law by penalty or subspace projection.
5. **No field operators, no second quantization:** all Hamiltonians are finite-dimensional matrices; all measured quantities are single-particle operator expectation values.

Figure 1: Summary of the first-quantized, two-register quantum simulation scheme for the Dirac–Maxwell system.

10 Quantum simulation for the first-quantized Yang–Mills system

The Yang–Mills system is treated in the same first-quantized spirit as the Dirac–Maxwell system: the gauge field $A_\mu^a(x)$ and the matter wave function $\psi_n(t)$ are c-number objects encoded as

quantum-state amplitudes. No field operators and no second quantization are used. The matter and gauge sectors are simulated on separate quantum registers and coupled through classical communication. The essential new feature compared with Maxwell is that the source-free Yang–Mills equation is already nonlinear, because of the A^3 and A^4 self-couplings.

10.1 Gauge-field register

Let N_s be the number of spatial lattice sites and $a = 1, \dots, N_c^2 - 1$ the adjoint color index ($N_c = 3$ for QCD, so $a = 1, \dots, 8$). The color-electric and color-magnetic fields are

$$E_i^a = F_{0i}^a, \quad B_i^a = -\frac{1}{2}\epsilon_{ijk}F_{jk}^a, \quad (157)$$

with

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c. \quad (158)$$

Define the color Riemann–Silberstein wave function

$$\mathbf{G}_i^{a,\pm}(\mathbf{x}, t) = \sqrt{\frac{\epsilon_0}{2}}(E_i^a(\mathbf{x}, t) \pm icB_i^a(\mathbf{x}, t)). \quad (159)$$

After imposing the color transversality conditions $D_i E_i^a = 0$ and $D_i B_i^a = 0$, the physical polarizations are encoded on register G as

$$|\Psi_g(t)\rangle = \sum_{n=1}^{N_s} \sum_{a=1}^{N_c^2-1} \sum_{\lambda=\pm} G_{n,\lambda}^a(t) |n\rangle \otimes |a\rangle \otimes |\lambda\rangle, \quad (160)$$

where $G_{n,\lambda}^a(t)$ are c-number amplitudes.

The free (Abelian-like) part of the gauge Hamiltonian is the finite-dimensional matrix

$$H_g^{(0)} = \pm c \left(\hat{\mathbf{S}}^{(1)} \cdot \hat{\mathbf{p}} \right) \otimes I_{\text{color}}, \quad (161)$$

acting on the color-neutral part of the wave function. The color-dependent self-coupling is generated by the matrices

$$(H_g^{(1)})_b^a = gf_{bc}^a A_{n,i}^c (\hat{S}_i^{(1)})_{\lambda\lambda'}, \quad (162)$$

which couple the color components through the structure constants. The full gauge Hamiltonian is

$$H_g = H_g^{(0)} + H_g^{(1)}[A] + H_g^{(2)}[A, A], \quad (163)$$

where $H_g^{(1)}$ is linear in the gauge potential and $H_g^{(2)}$ is quadratic, corresponding to the A^3 and A^4 vertices. The gauge wave function evolves as

$$i\hbar \partial_t \mathbf{G}^\pm = H_g[A] \mathbf{G}^\pm, \quad (164)$$

implemented by Trotterized unitary gates on register G . The Abelian hopping is realized by controlled exchange gates; the color mixing f^{abc} is realized by controlled rotations conditioned on the color register; the A^3 and A^4 terms are realized by multi-qubit controlled gates with coefficients supplied by the current gauge potential $A^{(k)}$.

10.2 Handling the non-Abelian self-coupling

Because $H_g^{(1)}$ and $H_g^{(2)}$ depend on A , the gauge equation is nonlinear already without matter. We treat this by an inner self-consistency loop on the gauge register alone. At each outer step k :

1. Use the current gauge potential $A^{(k)}$ to build the linearized Hamiltonian

$$H_g^{(k)} = H_g^{(0)} + H_g^{(1)}[A^{(k)}] + H_g^{(2)}[A^{(k)}, A^{(k)}]. \quad (165)$$

2. Evolve $\mathbf{G}^\pm$ unitarily with $e^{-iH_g^{(k)}\Delta t/\hbar}$ on register G .
3. Measure the color fields E_i^a, B_i^a , and reconstruct $A^{(k+1)}$ by solving the color Gauss law and the Bianchi identity on the lattice.
4. Repeat the inner loop until $A^{(k+1)}$ converges to $A^{(k)}$ within tolerance.

This is a quantum–classical self-consistency loop entirely within the gauge sector. It does not require second quantization: A_μ^a remains a c-number background, and $\mathbf{G}^\pm$ remains a first-quantized wave function.

10.3 Matter register and color current

The quark wave function at site n is a multi-component spinor

$$\psi_n(t) \in \mathbb{C}^{4N_f N_c}, \quad (166)$$

with N_f flavours, N_c colors, and four Dirac components. It is encoded on register E as

$$|\Psi_q(t)\rangle = \sum_{n=1}^{N_s} \sum_{A=1}^{4N_f N_c} \psi_{n,A}(t) |n\rangle \otimes |A\rangle. \quad (167)$$

Given the classical background $A_\mu^{(k)}$, the quark Hamiltonian is the Hermitian matrix

$$H_q[A^{(k)}] = \sum_n \beta m c^2 \otimes |n\rangle\langle n| + \sum_{\langle n,m \rangle} (-i\hbar c \boldsymbol{\alpha} \cdot \hat{\mathbf{e}}_{nm}) e^{ieA_{nm}^{(k)}/\hbar} \otimes |n\rangle\langle m| + \sum_n eA_{0,n}^{(k)} I \otimes |n\rangle\langle n|, \quad (168)$$

where $A_{nm}^{(k)} = A_\mu^{a,(k)} T^a$ is the color link variable in the fundamental representation. The quark wave function evolves as

$$i\hbar \partial_t \Psi_q = H_q[A^{(k)}] \Psi_q, \quad (169)$$

implemented by Trotterized gates on register E . The color current is the c-number bilinear

$$j_n^{a,\mu} = g \sum_{A,B=1}^{4N_f N_c} \psi_{n,A}^*(t) (\gamma^0 \gamma^\mu T^a)_{AB} \psi_{n,B}(t), \quad (170)$$

obtained as the expectation value of the single-particle operator

$$\hat{j}_n^{a,\mu} = g \gamma^0 \gamma^\mu T^a \otimes |n\rangle\langle n|, \quad (171)$$

which is a finite-dimensional matrix, not a field operator.

10.4 Classical communication and full protocol

The two registers are evolved separately and coupled through classical communication. At step k :

1. **Gauge self-consistency:** evolve the gauge register G with $e^{-iH_g[A^{(k)}]\Delta t/\hbar}$; measure E_i^a, B_i^a ; reconstruct $A^{(k+1)}$; iterate until convergence.
2. **Quark evolution:** evolve the quark register E with $e^{-iH_q[A^{(k)}]\Delta t/\hbar}$.

3. **color current:** measure $j^{a,\mu,(k+1)}$ on E .

4. **Gauge response:** add the source response to the gauge wave function by a controlled displacement

$$|\mathbf{G}^\pm\rangle \mapsto |\mathbf{G}^\pm + \mathbf{G}_p^\pm\rangle, \quad (172)$$

where $\mathbf{G}_p^\pm$ is the particular solution of the linearized inhomogeneous Yang–Mills equation driven by $j^{a,\mu,(k+1)}$.

5. Repeat for $k = 0, \dots, K - 1$.

The color Gauss law

$$(D_i E_i^a - \rho^a) |\Psi_{\text{phys}}\rangle = 0 \quad (173)$$

is enforced by a penalty term in the Hamiltonian or by restricting to the constrained subspace. The color transversality conditions are maintained by encoding only the physical polarizations and projecting the initial state.

10.5 Resources and limitations

The gauge register requires $N_s \times (N_c^2 - 1) \times 2$ qubits for two polarizations and $N_c^2 - 1$ colors. The quark register requires $N_s \times (4N_f N_c)$ qubits (or fewer with binary encoding). Each Trotter step uses $O(N_s N_c^2)$ two-qubit gates for the Abelian part and $O(N_s N_c^3)$ multi-qubit gates for the color mixing and self-coupling. The total gate count scales as $O(K N_s N_c^3)$ with $K = t/\Delta t$. The scheme is first-quantized: quark number is fixed, and quark–antiquark pair creation, gluon-number fluctuations, and confinement effects beyond the mean-field level are not described. The nonlinearity of the full Yang–Mills system appears through the iterative classical communication and the inner gauge self-consistency loop; each individual quantum evolution is linear and unitary.

First-quantized Yang–Mills simulation protocol.

1. **Gauge register G :** encode transverse color Riemann–Silberstein wave function $\mathbf{G}^{a,\pm}$ as $|\Psi_g\rangle = \sum_{n,a,\lambda} G_{n,\lambda}^a |n, a, \lambda\rangle$. Evolve with $e^{-iH_g[A^{(k)}]\Delta t/\hbar}$, where H_g contains the Abelian part and the A^3, A^4 self-couplings.
2. **Gauge self-consistency:** measure E_i^a, B_i^a ; reconstruct $A^{(k+1)}$; iterate until convergence.
3. **Quark register E :** encode $\Psi_q \in \mathbb{C}^{4N_f N_c N_s}$ as $|\Psi_q\rangle = \sum_{n,A} \psi_{n,A} |n, A\rangle$. Evolve with $e^{-iH_q[A^{(k)}]\Delta t/\hbar}$.
4. **Classical communication:** measure color current $j^{a,\mu}$ on E ; add gauge source response by controlled displacement on G ; update A .
5. **Constraints:** color Gauss law by penalty or subspace projection; color transversality by encoding only physical polarizations.
6. **No field operators, no second quantization:** all Hamiltonians are finite-dimensional matrices; all measured quantities are single-particle operator expectation values.

Figure 2: Summary of the first-quantized, two-register quantum simulation scheme for the Yang–Mills system.

11 Quantum simulation for the Einstein field equations

We formulate the simulation in the same first-quantized spirit as the Dirac–Maxwell and Yang–Mills cases. The graviton field $h_{\mu\nu}(x)$ and the matter wave function $\psi_n(t)$ are c-number objects encoded as quantum-state amplitudes. No field operators and no second quantization are

used. The graviton and matter sectors are simulated on separate quantum registers and coupled through classical communication. The nonlinear self-coupling of the gravitational field is handled by an inner self-consistency loop with a controlled truncation.

11.1 Graviton register

Work in the harmonic gauge $\partial^\mu \bar{h}_{\mu\nu} = 0$ with

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \quad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \quad \kappa = \sqrt{16\pi G}. \quad (174)$$

The physical degrees of freedom are the transverse-traceless modes h_{ij}^{TT} , with two helicities $\lambda = \pm 2$. At each lattice site we encode them on register G as

$$|\Psi_g(t)\rangle = \sum_{n=1}^{N_s} \sum_{\lambda=\pm} h_{n,\lambda}(t) |n\rangle \otimes |\lambda\rangle, \quad (175)$$

where $h_{n,\lambda}(t)$ are c-number amplitudes. The free Hamiltonian is the finite-dimensional matrix

$$H_g^{(0)} = \pm c \left(\hat{\mathbf{S}}^{(2)} \cdot \hat{\mathbf{p}} \right), \quad (176)$$

restricted to the TT subspace, with $\hat{\mathbf{S}}^{(2)}$ the spin-2 matrices and $\hat{\mathbf{p}}$ the lattice momentum operator. It does not contain the wave function h .

The free graviton wave function evolves as

$$i\hbar \partial_t h = H_g^{(0)} h, \quad (177)$$

implemented by Trotterized unitary gates $e^{-iH_g^{(0)}\Delta t/\hbar}$ on register G . The two polarizations propagate independently; the lattice hopping is realized by controlled exchange gates and single-qubit phase gates.

11.2 Gravitational self-coupling and truncation

The full nonlinear equation is

$$\square \bar{h}_{\mu\nu} = -16\pi G \tau_{\mu\nu}, \quad \tau_{\mu\nu} = T_{\mu\nu} + t_{\mu\nu}[h], \quad (178)$$

where $T_{\mu\nu}$ is the matter energy-momentum and $t_{\mu\nu}[h]$ is the gravitational self-energy. Expanding in κh ,

$$t_{\mu\nu}[h] = t_{\mu\nu}^{(2)}[h^2] + \kappa t_{\mu\nu}^{(3)}[h^3] + \kappa^2 t_{\mu\nu}^{(4)}[h^4] + \dots \quad (179)$$

We truncate at order n_{max} :

$$t_{\mu\nu}^{\text{trunc}} = \sum_{n=2}^{n_{\text{max}}} \kappa^{n-2} t_{\mu\nu}^{(n)}. \quad (180)$$

Given the current background $h^{(k)}$, the linearized Hamiltonian is

$$H_g^{(k)} = H_g^{(0)} + H_g^{(1)}[h^{(k)}] + H_g^{(2)}[h^{(k)}, h^{(k)}] + \dots + H_g^{(n_{\text{max}})}[(h^{(k)})^{n_{\text{max}}}], \quad (181)$$

and the graviton wave function evolves as

$$i\hbar \partial_t h = H_g^{(k)} h. \quad (182)$$

The nonlinear terms are realized by multi-qubit controlled gates with coefficients supplied by the current background $h^{(k)}$. Because $t_{\mu\nu}[h]$ depends on h , the graviton sector alone requires an inner self-consistency loop: evolve h , measure h and its derivatives, rebuild $H_g^{(k)}$, and iterate until convergence.

11.3 Matter register

The matter wave function is encoded on register E as

$$|\Psi_m(t)\rangle = \sum_{n=1}^{N_s} \sum_A \psi_{n,A}(t) |n\rangle \otimes |A\rangle, \quad (183)$$

where A labels the internal components (e.g. two components for a scalar field, four for a Dirac field). Given the classical metric background

$$g_{\mu\nu}^{(k)} = \eta_{\mu\nu} + \kappa h_{\mu\nu}^{(k)}, \quad (184)$$

the matter Hamiltonian $H_m[g^{(k)}]$ is a finite-dimensional Hermitian matrix acting on the matter wave function. For a scalar field,

$$H_m[g] = \int d^3x \frac{1}{2\sqrt{-g}} \left[\pi^2 + g^{ij} \partial_i \phi \partial_j \phi + m^2 \phi^2 \right], \quad (185)$$

discretized on the lattice. The matter wave function evolves as

$$i\hbar \partial_t \Psi_m = H_m[g^{(k)}] \Psi_m, \quad (186)$$

implemented by Trotterized gates on register E .

The matter energy-momentum tensor is the c-number bilinear

$$T_{\mu\nu,n} = (\text{quadratic form in } \psi_n \text{ and its derivatives}), \quad (187)$$

obtained as the expectation value of a single-particle operator on register E . It is not a field operator.

11.4 Classical communication and full protocol

The two registers are evolved separately and coupled through classical communication. At step k :

1. **Graviton self-consistency:** use $h^{(k)}$ to build $H_g^{(k)}$ with truncation order $n_{\max}$; evolve the graviton register G ; measure $h^{(k+1)}$ and its derivatives; iterate until convergence.
2. **Matter evolution:** evolve the matter register E with $H_m[g^{(k)}]$.
3. **Matter source:** measure $T_{\mu\nu}^{(k+1)}$ on E .
4. **Graviton source response:** form the total source

$$\tau_{\mu\nu}^{(k+1)} = T_{\mu\nu}^{(k+1)} + t_{\mu\nu}^{\text{trunc}}[h^{(k)}], \quad (188)$$

solve the linearized equation

$$\square \bar{h}_{\mu\nu} = -16\pi G \tau_{\mu\nu}^{(k+1)}, \quad (189)$$

and add the particular solution $\bar{h}_p^{(k+1)}$ to the graviton wave function by a controlled displacement

$$|h\rangle \longmapsto |h + h_p\rangle. \quad (190)$$

5. **Metric update:** measure $h^{(k+1)}$, reconstruct $g_{\mu\nu}^{(k+1)}$, and pass it to the matter register.
6. Repeat for $k = 0, \dots, K - 1$.

11.5 Constraints and gauge fixing

The physical states are restricted by:

- **TT condition:** only the two transverse-traceless polarizations are encoded, so $\partial_i h_{ij}^{\text{TT}} = 0$ and $h_{ii}^{\text{TT}} = 0$ hold by construction.
- **Harmonic gauge:** enforced by a penalty term

$$H_{\text{gauge}} = \mu \sum_n \left(\partial^\mu \bar{h}_{\mu\nu,n} \right)^2. \quad (191)$$

- **Hamiltonian and momentum constraints:**

$$\mathcal{H} = 0, \quad \mathcal{H}_i = 0, \quad (192)$$

enforced by a penalty term

$$H_{\text{constraint}} = \nu \sum_n \left(\mathcal{H}_n^2 + \mathcal{H}_{i,n} \mathcal{H}_n^i \right), \quad (193)$$

or by projecting onto the constrained subspace after each Trotter step.

11.6 Truncation error

Define the dimensionless expansion parameter

$$\varepsilon \equiv |\kappa h_{\mu\nu}|. \quad (194)$$

Truncating the self-coupling at order n_{max} gives

$$\delta_{\text{trunc}} \sim \varepsilon^{n_{\text{max}}+1}. \quad (195)$$

Numerical estimates:

Regime	ε	n_{max}	δ_{trunc}
Solar system	10^{-6}	2	$\sim 10^{-18}$
Gravitational waves	$h \sim 10^{-21}$	1 (linear)	$\sim 10^{-62}$
Neutron star surface	0.1–0.3	4	$\sim 10^{-4}$ – 10^{-3}
Black-hole merger	~ 1	—	perturbative expansion fails

In weak-field regimes the truncation error is far below lattice and Trotter errors and is not the bottleneck. For neutron-star-strength fields, $n_{\text{max}} = 4$ – 6 is needed for 10^{-4} – 10^{-6} accuracy. For strong fields ($\varepsilon \sim 1$) the perturbative truncation is invalid and a nonperturbative scheme is required.

11.7 Comparison with Maxwell and Yang–Mills

11.8 Resources and limitations

The graviton register requires N_s qubits for the two polarizations. The matter register requires $N_s \times n_{\text{comp}}$ qubits, where n_{comp} is the number of internal components. Each Trotter step uses $O(N_s)$ two-qubit gates for the free graviton evolution and $O(N_s n_{\text{max}})$ multi-qubit gates for the truncated self-coupling. The total gate count scales as $O(K N_s n_{\text{max}})$ with $K = t/\Delta t$. The constraint penalty terms add $O(N_s)$ two-qubit gates per Trotter step.

The scheme is first-quantized: graviton number is fixed, and graviton-number fluctuations, graviton–matter entanglement beyond the mean-field level, and nonperturbative strong-field effects are not described. The nonlinearity of the full Einstein equations appears through the iterative classical communication and the inner graviton self-consistency loop; each individual quantum evolution is linear and unitary. The perturbative expansion $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ restricts the scheme to weak and intermediate fields.

	Maxwell	Yang–Mills	Gravity
Wave function	$\mathbf{F}^\pm$ (spin-1)	$\mathbf{G}^{a,\pm}$ (colored spin-1)	h_{ij}^{TT} (spin-2)
Free H	$\pm c \hat{\mathbf{S}}^{(1)} \cdot \hat{\mathbf{p}}$	$H_g^{(0)} + H_g^{(1)}[A] + H_g^{(2)}[A, A]$	$H_g^{(0)} + H_g^{(1)}[h] + \dots$
Nonlinearity	none	terminates at A^4	infinite order
Self-coupling	absent	present	present
Constraints	Gauss law	color Gauss law	Hamiltonian + momentum
Gauge group	$U(1)$	$SU(N_c)$	diffeomorphisms
Source	$j^\mu[\psi]$	$j^{a,\mu}[\psi]$	$T_{\mu\nu}[\psi] + t_{\mu\nu}[h]$

Table 1: Comparison of the first-quantized Maxwell, Yang–Mills, and gravity simulation schemes.

First-quantized gravity simulation protocol.

1. **Graviton register G :** encode h_{ij}^{TT} as $|\Psi_g\rangle = \sum_{n,\lambda} h_{n,\lambda} |n, \lambda\rangle$. Evolve with $e^{-iH_g^{(k)} \Delta t/\hbar}$, where $H_g^{(k)}$ contains the free spin-2 part and the truncated self-coupling.
2. **Graviton self-consistency:** measure h and its derivatives; rebuild $H_g^{(k)}$; iterate until convergence.
3. **Matter register E :** encode Ψ_m as $|\Psi_m\rangle = \sum_{n,A} \psi_{n,A} |n, A\rangle$. Evolve with $e^{-iH_m[g^{(k)}] \Delta t/\hbar}$.
4. **Classical communication:** measure $T_{\mu\nu}$ on E ; form $\tau_{\mu\nu} = T_{\mu\nu} + t_{\mu\nu}^{\text{trunc}}$; add the particular solution h_p by controlled displacement on G ; reconstruct $g_{\mu\nu}$.
5. **Constraints:** TT encoding; harmonic gauge and Hamiltonian/momentum constraints by penalty or subspace projection.
6. **No field operators, no second quantization:** all Hamiltonians are finite-dimensional matrices; all measured quantities are single-particle operator expectation values.

Figure 3: Summary of the first-quantized, two-register quantum simulation scheme for the Einstein field equations.

12 Quantum field: second quantization

The second quantization refers to the process of promoting Hilbert space to Fock space to account for fluctuation of particle numbers. The wavefunction in Hilbert space has to be promoted to field operators acting on Fock space. A new wavefunction in Fock space will emerge to describe the new evolution. As far as we know, Einstein’s field theory is not second quantized, and it is still a major issue in modern study of quantum gravity.

The quantum simulation of quantum field theories is a major goal of quantum computing. In this section, we lay out a scheme for the sine–Gordon field theory.

In the present context, the sine–Gordon model is most useful as a quantum field theory in one spatial dimension and one time dimension [3]. It provides a low-energy description of several interacting quantum systems, including spin chains, interacting fermions, and bosons in periodic potentials.

The scalar field in this description is a collective variable. It need not represent an elementary spin-zero particle. In particular, a scalar field theory can describe the long-wavelength dynamics of a chain whose microscopic degrees of freedom are spin-1/2 operators.

12.0.1 The Luttinger-liquid starting point

Many gapless one-dimensional quantum systems have a low-energy Hamiltonian of the form

$$H_{\text{LL}} = \frac{v}{2\pi} \int dx \left[K(\partial_x \hat{\theta})^2 + \frac{1}{K}(\partial_x \hat{\phi})^2 \right]. \quad (196)$$

Here:

- v is the velocity of the collective excitations;
- K is the Luttinger parameter, which controls the scaling of physical correlation functions;
- $\hat{\phi}$ and $\hat{\theta}$ are conjugate compact bosonic fields.

One conventional normalization is

$$[\hat{\phi}(x), \partial_{x'} \hat{\theta}(x')] = i\pi \delta(x - x'). \quad (197)$$

The Hamiltonian (196) is quadratic in these collective fields. Nevertheless, the original particles or spins are not generally independent.

Physical spin and particle operators involve derivatives and vertex operators such as

$$e^{ip\hat{\phi}(x)}, \quad e^{iq\hat{\theta}(x)}. \quad (198)$$

Their correlation functions exhibit interaction-dependent power laws. A Gaussian collective-field description can therefore encode substantial quantum correlations in the microscopic variables.

This is an important distinction: a quadratic effective Hamiltonian does not imply that the microscopic quantum state is uncorrelated.

12.0.2 From a Luttinger liquid to sine–Gordon theory

The microscopic symmetries can allow additional operators. A common example is

$$H_{\text{SG}} = H_{\text{LL}} + \lambda \int dx \cos(p\hat{\phi}(x)), \quad (199)$$

where the vertex operator is understood with an ultraviolet cutoff. The allowed value of p depends on the compactification of the field and on the microscopic symmetries.

Such a term can represent a lattice scattering process, a commensurate perturbation, or a perturbation of a spin chain. Other systems instead allow a cosine of the dual field $\hat{\theta}$.

After eliminating the conjugate field in the action, the corresponding real-time Lagrangian density is

$$\mathcal{L}_{\text{SG}} = \frac{1}{2\pi K} \left[\frac{1}{v} (\partial_t \hat{\phi})^2 - v (\partial_x \hat{\phi})^2 \right] - \lambda \cos(p\hat{\phi}). \quad (200)$$

The sign of λ selects which minima are preferred.

The classical stationary equation of this action is

$$\frac{1}{v^2} \partial_t^2 \phi - \partial_x^2 \phi - \frac{\pi K \lambda p}{v} \sin(p\phi) = 0. \quad (201)$$

A shift of ϕ can convert this into the familiar sine–Gordon equation with the opposite sign of the sine term.

However, the quantum theory is defined by quantizing (199) or by its quantum path integral. It is not exhausted by solving the classical nonlinear equation (201).

12.0.3 When does the cosine change the low-energy physics?

With the normalization in (196), the scaling dimension of $\cos(p\hat{\phi})$ at the Gaussian fixed point is

$$\Delta_p = \frac{p^2 K}{4}. \quad (202)$$

This gives a useful first test:

- If $\Delta_p > 2$, a weak cosine perturbation is irrelevant: its effect decreases toward low energies.
- If $\Delta_p < 2$, it is relevant: it grows and can pin the field, producing a gap.
- If $\Delta_p = 2$, it is marginal at this order. Its fate requires the coupled renormalization-group flow.

The field can interpolate between neighboring minima of the cosine potential. Such configurations are solitons and antisolitons. In appropriate parameter regimes, the quantum sine–Gordon spectrum also contains their bound states, called breathers.

Depending on the microscopic realization, these excitations can represent domain walls or other collective excitations. The appearance of a scalar field does not remove the underlying many-body interpretation.

12.0.4 Example: the spin-1/2 XXZ chain

Consider the antiferromagnetic XXZ chain at zero magnetic field:

$$H_{\text{XXZ}} = J \sum_j \left[S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \right], \quad J > 0. \quad (203)$$

The transverse exchange terms generate quantum spin fluctuations. This is not simply an Ising model.

For

$$-1 < \Delta < 1, \quad (204)$$

the low-energy theory is a Luttinger liquid. In a standard bosonization convention, a leading translation-invariant lattice perturbation is

$$H_{\text{pert}} = \lambda_4 \int dx \cos(4\hat{\phi}). \quad (205)$$

Its Gaussian scaling dimension is $4K$.

At the isotropic antiferromagnetic point $\Delta = 1$, $K = 1/2$, and this operator is marginal. The flow near this point describes the transition from the gapless regime to the easy-axis gapped phase. For $\Delta > 1$, the system has a gap and Néel order at zero temperature.

Dimerization or a staggered field can allow different vertex operators and generate other gapped phases. Thus sine–Gordon theory describes a family of non-Ising quantum spin problems, although not every one-dimensional spin system belongs to this family.

The microscopic model determines which cosine operators are allowed, their coefficients, and the physical meaning of the resulting excitations.

The sine–Gordon model can be simulated by a spin chain, or, a collection of qubits with proper Hamiltonian. Ground states, correlation functions, and dynamics can be easily simulated.

12.0.5 Nonlinearity does not imply the neglect of entanglement

The quantum state of the sine–Gordon theory obeys

$$i\partial_t |\Psi(t)\rangle = \hat{H}_{\text{SG}} |\Psi(t)\rangle. \quad (206)$$

This evolution is linear in the full state.

The interaction is nonlinear in the collective field operator because the Hamiltonian contains $\cos(p\hat{\phi})$. Its quantum dynamics can generate and retain entanglement and nontrivial correlations.

In particular,

$$\langle \sin(p\hat{\phi}) \rangle \neq \sin(p\langle \hat{\phi} \rangle) \quad (207)$$

in general. Replacing the left-hand side by the right-hand side would introduce an additional mean-field approximation; it is not part of the definition of quantum sine-Gordon theory.

The lesson is therefore twofold:

1. A nonlinear low-energy quantum field theory can retain strong correlations.
2. A quadratic collective-field theory, such as a Luttinger liquid, can already describe a highly nontrivial microscopic quantum state.

Nonlinearity of a field equation and entanglement of a quantum state are distinct properties. Neither can be used as a universal diagnostic for the other.

13 Conclusion and Outlook

We have examined the simulation of relativistic quantum mechanics from a first-quantized perspective, with emphasis on the structure of spin-momentum coupling, the origin of nonlinearity, and the practical obstacles that arise when bosonic and fermionic degrees of freedom are treated on quantum hardware.

Spin-momentum coupling and nonlinearity. For spin-0, $1/2$, 1, and 2, the free relativistic wave equations share a common spin-momentum structure, and the transition from linear to nonlinear dynamics has a uniform interpretation: nonlinearity emerges when part of the entangled quantum system is replaced by a classical background. In QED and QCD this appears through minimal coupling and self-interaction; in gravity it appears through the self-coupling of the spin-2 field, whose bootstrap completion is the Einstein-Hilbert action. In every case the underlying quantum evolution remains linear and unitary; the nonlinear classical equations are effective descriptions obtained after tracing out or averaging over the neglected degrees of freedom.

First-quantized simulation scheme. We formulated a first-quantized, register-separated simulation strategy in which matter and gauge wave functions are encoded on independent quantum registers, evolve unitarily under finite-dimensional Hermitian matrices, and exchange sources and backgrounds through classical communication. Source terms are absorbed by particular solutions added through controlled displacements, so that every quantum step remains linear. Constraints and gauge conditions are enforced by penalty terms or subspace projection, and the infinite-order gravitational self-coupling is truncated with controlled error. This scheme avoids the Fock-space growth of bosons and the nonlocality of higher-dimensional Jordan-Wigner maps, at the cost of fixed particle number and mean-field coupling.

Bosons and fermions in higher dimensions. Exact Jordan-Wigner and bosonization techniques are confined to one spatial dimension. The one-dimensional chain sine-Gordon $\leftrightarrow$ free boson $\leftrightarrow$ free Dirac fermion is mathematically reliable and provides benchmark platforms, but a single bosonic field carries only one chirality pair; a spinful electron requires two bosonic fields, i.e. spin-charge separation, which is itself one-dimensional. In $2+1$ and $3+1$ dimensions the naive Jordan-Wigner map destroys locality, and exact bosonization requires dynamical lattice gauge fields. No mature, dimension-independent, fully quantum treatment of interacting bosonic and fermionic degrees of freedom currently exists.

QED and QCD phenomenology. For QED, single-particle phenomena such as Zitterbewegung, Klein tunneling, spin-orbit coupling, and strong-field wave-packet dynamics are accessible to first-quantized simulation and serve as algorithm benchmarks, while genuine multi-particle phenomena—Schwinger pair production, spontaneous emission, vacuum birefringence, and electron-photon entanglement—require second quantization. For QCD, lattice gauge theory provides a natural spin-chain-like formulation in which gluon self-coupling appears as a local interaction, but confinement, chiral symmetry breaking, and real-time transport at finite density remain beyond current quantum resources. The first-quantized scheme presented here captures

the self-consistent mean-field sector of these theories but does not resolve their genuinely many-body content.

Quantum gravity. The gravitational case is the most severe. The self-coupling is of infinite order and must be truncated; the expansion parameter κh is tiny in the solar system and for gravitational waves, where truncation error is far below lattice and Trotter errors, but reaches order unity at neutron-star surfaces and black-hole mergers, where the perturbative scheme fails. The Hamiltonian and momentum constraints, the choice of gauge, and the non-localizability of gravitational energy all require careful treatment. Whether a quantum gravity theory must reduce to Einstein’s equations at low energy is not a logical necessity: additional light degrees of freedom, Lorentz violation, graviton mass, higher-curvature terms, nonlocality, or a non-geometric ultraviolet completion can all modify the low-energy limit, although observations constrain such deviations very tightly. Finsler and other non-Riemannian geometries offer phenomenological frameworks for parametrizing these deviations but are not established ultraviolet completions.

Outlook. The natural next steps are threefold. First, extend the register-separated scheme from fixed particle number to truncated Fock spaces, so that pair production, spontaneous emission, and genuine entanglement dynamics become accessible. Second, develop dimension-independent encodings that preserve locality without introducing expensive gauge-theoretic bosonization. Third, quantify the resource requirements for the strong-field and strong-curvature regimes where classical methods are least reliable, and identify the smallest systems in which quantum simulation can outperform classical numerics.

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